June 1994

arXiv:0803.3204v1 [gr-qc] 21 Mar 2008

The semiclassical stability of de Sitter spacetime

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Abstract

de Sitter spacetime and Bunch-Davies vacuum are a solution to the semiclassical Einstein-Schrödinger equations describing the evolution of spacetime geometry and a massive scalar quantum field with arbitrary coupling to curvature. The stability of this solution is proven by calculating the renormalized energy momentum tensor expectation value for small spatially homogeneous deviations from the de Sitter – Bunch-Davies system and solving the linearized backreaction problem. A renormalization scheme is developed. All momentum integrations are carried out analytically. The general solution is given in terms of its Laplace transform. It contains only two artificial instabilities: a constant gauge mode and an instability on the Planck time scale lying outside of the scope of our semiclassical theory.

47 pages, 3 figures

1 Introduction

The de Sitter spacetime [1] is of great theoretical as well as cosmological interest. The former arises due to its high degree of symmetry: with 10 Killing vector fields its isometry group O(4,1) has the same dimension as the Poincaré group of Minkowski spacetime and therefore the maximum dimension the symmetry group of a four dimensional spacetime can have at all. Just this fact makes a lot of calculations of quantum field theory feasible in the de Sitter spacetime.
The cosmological interest stems from the exponential growth of the scale factor in the spatially flat (k=0) Friedmann-Robertson-Walker parametrization of de Sitter spacetime solving some basic problems of the standard cosmology in inflationary universe models [2].
In absence of a consistent quantum theory of gravity one usually works within a semiclassical framework, where the gravitational field is treated as a classical background field and only the matter fields are quantized. This is justified as long as all relevant inverse time and length scales are small compared to the Planck scale[1], so that quantum gravity effects are expected to be small.
Since the work of Schwinger it is known that the quantum fluctuations of a charged matter field in an electromagnetic background field can lead to the production of particle-antiparticle pairs. The same applies to the gravitational background field and is known as the Hawking effect. From this observation the conjecture and also some claims arose in the literature (see for example [3]-[4]), that in the presence of a scalar quantum field like in most inflationary scenarios the de Sitter spacetime might be unstable due to particle production and should decay in some sense by itself towards a flat spacetime.
There is no unique observer-independent particle-antiparticle concept in a general curved spacetime and different approaches to particle production involving different approximations led to different answers for this stability question.

[1] in natural units $\hbar = c = 1$

In reference [5] on the contrary this question is adressed in a rather clear and reliable manner (see below) based on the energy momentum tensor, an observer-independent physical quantity. Unfortunately no sensible results were obtained due to technical problems, on which we will comment later. In a further publication [6] the main problems were not eliminated. Nevertheless their approach is promising and will be adopted as the starting point for the present investigation.

Within the semiclassical theory the evolution of spacetime geometry is governed by the Einstein equations containing the expectation value of the energy momentum tensor as the source on the right-hand side, whereas the quantum state of the scalar field has to obey the Schrödinger equation, which in turn depends on the spacetime metric:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R + \Lambda\, g_{\mu\nu} &= 8\pi G_N \, \langle\Psi| T_{\mu\nu} |\Psi\rangle \\ i\,\partial_t\,|\Psi\rangle &= \hat{H}_g\,|\Psi\rangle \end{aligned} \tag{1}$$

The de Sitter spacetime and the Bunch-Davies vacuum, a special state of the quantum field, are a solution to this semiclassical system of coupled equations. In order to investigate the stability of this solution against small fluctuations of the gravitational field and of the quantum state we will linearize the equations (1) around the de Sitter – Bunch-Davies solution. This linearization is the only approximation appearing within this work. The linearized Einstein-Schrödinger equations will be solved completely and the general solution will be analyzed with respect to instabilities.

In the course of this work another publication [7] on the same subject appeared also based on reference [5]. We will reach the same conclusions as reference [7] but in a more direct way, because in the calculation of the energy momentum tensor we will execute the momentum integrations analytically, so that the result is suited for a numerical analysis. Furthermore a general coupling of the scalar quantum field to curvature will be allowed.

The paper is organized as follows: section 2 gives a brief review of some important

results from reference [8], whereas section 3, the main part of this work, contains the calculation of the energy momentum tensor expectation value and the isolation of its divergencies. The linear stability analysis is performed in section 4. Some mathematical tools are collected in the appendix.

2 Schrödinger picture field theory in k=0 Friedmann-Robertson-Walker spacetimes

This section deals with the quantum theory of a free, massive scalar field in a spatially flat (k=0) Friedmann-Robertson-Walker (FRW) spacetime, and it is a brief summary of some results from reference [8].

Since dimensional regularization will be applied later on, we work in d+1 spacetime dimensions and on flat d-dimensional spacelike hyperplanes with coordinates $\vec{x} = (x_1, \ldots, x_d)$. The maximum symmetry of these hyperplanes will greatly simplify the following calculations. The k=0 FRW-metric has the form

$$ds^2 = dt^2 - a^2(t)\, d\vec{x} \cdot d\vec{x} = g_{\mu\nu} dx^\mu dx^\nu, \tag{2}$$

where $a(t)$ is the FRW scale factor. In terms of the Hubble function $H(t) := \dot{a}(t)/a(t)$, $\dot{a} := \partial_t a$, one obtains for the Ricci tensor

$$R_{00} = -d(\dot{H} + H^2)\ , \qquad R_{ij} = a^2(\dot{H} + dH^2)\delta_{ij}\ , \qquad R_{0i} = 0 \tag{3}$$

and for the curvature scalar $R = -d(2\dot{H} + (d+1)H^2)$.

A scalar field ϕ of mass m is supposed to interact only with the classical gravitational field $g_{\mu\nu}$ and may have an arbitrary coupling ξ to the curvature scalar R. Its action reads ($\sqrt{g} := a^d$)

$$S = \int d^{d+1}x\, \sqrt{g}\, \frac{1}{2}\, (g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - (m^2 - \xi R)\phi^2). \tag{4}$$

The quantum theory is formulated in the Schrödinger picture using a wave functional to represent the quantum state. This shows very clearly the real time evolution character of our analysis. Then the quantum operators are acting in the Fock

space on wave functionals $\Psi[\phi;t]$.

The spacelike derivatives contained in the Hamiltonian can be dealt with by performing a Fourier transform ($d\tilde{k} := d^dk/(2\pi)^{d/2}$, $\alpha^*(\vec{k}) = \alpha(-\vec{k})$):

$$\phi(\vec{x}) \;=\; \int d\tilde{k}\, e^{i\vec{k}\vec{x}}\, \alpha(\vec{k}) \;, \qquad \frac{\delta}{\delta\phi(\vec{x})} \;=\; \int d\tilde{k}\, e^{-i\vec{k}\vec{x}}\, \frac{\delta}{\delta\alpha(\vec{k})} \;, \tag{5}$$

and the Schrödinger equation resp. the Hamiltonian takes the following form:

$$\begin{aligned} i\,\partial_t\,\Psi[\alpha;t] &= \hat{H}\,\Psi[\alpha;t] \qquad (6)\\ \hat{H} &= \frac{1}{2}\int d^dk \Big(-\frac{1}{\sqrt{g}}\frac{\delta^2}{\delta\alpha(\vec{k})\delta\alpha(-\vec{k})} + \sqrt{g}(a^{-2}\vec{k}^2{+}m^2{-}\xi R)\alpha(\vec{k})\alpha(-\vec{k})\Big) \end{aligned}$$

For a curved spacetime without an everywhere timelike Killing vector field no unique Fock vacuum does exist. Rather there is a whole class of Fock vacua, which can all be represented by Gaussian wave functionals. We want our $\Psi[\alpha;t]$ to be a member of this class. Furthermore we require the quantum state not to break spontaneously the symmetries of the k=0 FRW metric (homogeneity and isotropy of the spacelike hyperplanes), which leads to the following wave functional parametrized by one function $A(k,t)$ (the inverse Gaussian width):

$$\Psi[\alpha;t] \;=\; N(t)\exp\Big(-\frac{1}{2}\int d^dk\, A(k,t)\,\alpha(\vec{k})\,\alpha(-\vec{k}) - i\Omega(t)\Big), \tag{7}$$

where $N(t)$ is a real normalization factor, $\Omega(t)$ a real phase and $k := |\vec{k}|$.

Substituting (7) in the Schrödinger equation (6) we get the equation of motion for $A(k,t)$:

$$i\,\dot{A}(k,t) \;=\; \frac{A^2(k,t)}{\sqrt{g(t)}} - \sqrt{g(t)}\,(a^{-2}(t)k^2{+}m^2{-}\xi R(t)) \tag{8}$$

This is Riccati's equation, and by the transformation $A(k,t) =: \sqrt{g}(\Gamma(k,t) + i\frac{d}{2}H(t))$ it takes the standard form

$$i\,\dot{\Gamma}(k,t) \;=\; \Gamma^2(k,t) + \frac{d^2}{4}H^2(t) + \frac{d}{2}\dot{H}(t) - (a^{-2}(t)k^2{+}m^2{-}\xi R(t)), \tag{9}$$

which can be converted by $\Gamma(k,t) =: -i\partial_t \ln u(k,t)$ into the linear equation

$$\ddot{u} \;-\; (\frac{d^2}{4}H^2 + \frac{d}{2}\dot{H} - (a^{-2}k^2{+}m^2{-}\xi R))\,u \;=\; 0 \quad . \tag{10}$$

From (8) the following equations for $A(k,t)$ can be derived, which are useful for the calculation of the energy momentum expectation value:

$$-\partial_t \left(\frac{1}{2\,\mathrm{Re}A}\right) \;=\; \frac{1}{\sqrt{g}}\,\frac{\mathrm{Im}A}{\mathrm{Re}A} \tag{11}$$

$$\frac{1}{g}\,\frac{|A|^2}{2\,\mathrm{Re}A} \;=\; (a^{-2}k^2+m^2-\xi R)\,\frac{1}{2\,\mathrm{Re}A} + \frac{1}{2\sqrt{g}}\,\partial_t\left(\sqrt{g}\partial_t\left(\frac{1}{2\,\mathrm{Re}A}\right)\right) \tag{12}$$

The energy momentum tensor acting as the source in the Einstein equations is defined as variational derivative of the matter action with respect to the metric tensor

$$T_{\mu\nu}(x) \;:=\; \frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta g^{\mu\nu}(x)}\;.$$

Due to the spatial symmetries the expectation value of the corresponding operator in the Gaussian state can be written as

$$\langle\Psi|(T^{\mu}{}_{\nu})|\Psi\rangle \;=\; \begin{pmatrix} \rho(t) & & & \\ & -p(t) & & \\ & & -p(t) & \\ & & & -p(t) \end{pmatrix}\;, \tag{13}$$

and the explicit calculation leads to the energy density ρ and pressure p in terms of the width $A(k,t)$:

$$\begin{aligned} \rho \;&=\; \frac{1}{2}\int \frac{\tilde{\tilde{dk}}}{2\,\mathrm{Re}A(k,t)}\Big(\frac{|A(k,t)|^2}{g} + a^{-2}k^2 + m^2 + 2\xi\,G_{00} - 2\xi\,\frac{dH}{\sqrt{g}}\,2\,\mathrm{Im}A(k,t)\Big) \\ p \;&=\; \frac{1}{2}\int \frac{\tilde{\tilde{dk}}}{2\,\mathrm{Re}A(k,t)}\Big(\frac{|A(k,t)|^2}{g} + (\frac{2}{d}-1)\,a^{-2}k^2 - m^2 + 2\xi\,a^{-2}G_{11} \\ &\qquad + 4\xi\,(a^{-2}k^2+m^2-\xi R) - 4\xi\,\frac{|A(k,t)|^2}{g} - 2\xi\,\frac{H}{\sqrt{g}}\,2\,\mathrm{Im}A(k,t)\Big) \end{aligned} \tag{14}$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R$ is the Einstein tensor, and $\tilde{\tilde{dk}} := d^dk/(2\pi)^d$.

Equation (8) shows, that $A(k,t)$ is of the order of k for large k. Hence the energy momentum expectation value (14) is quartic divergent and has to be renormalized. These ultraviolet divergencies are due to the behaviour of the wave functional for field configurations of high energy and momenta (large k) or resp. for small

distances and are connected to the local geometry of the underlying spacetime manifold. For this reason they should be proportional to local geometric tensors which can be absorbed into the gravitational part of the Einstein equations. Thus the divergencies can be removed by a renormalization of the physical parameters in the Einstein equations (cosmological constant, Newton's constant and additional parameters mentioned below). Fortunately the divergencies of the energy momentum expectation value can be calculated as a local functional of a general metric tensor by means of the De Witt-Schwinger-Christensen expansion. It turns out that in the Einstein equations one has to admit the geometrical tensors $H_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$, which are the metric variations $1/\sqrt{g}\ \delta/\delta g^{\mu\nu}$ of the functionals $\int d^{d+1}x\ \sqrt{g}\, R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma}$, $\int d^{d+1}x\ \sqrt{g}\, R^2$ and $\int d^{d+1}x\ \sqrt{g}\, R^{\alpha\beta} R_{\alpha\beta}$. Their renormalized coefficients have to be regarded as additional physical parameters of the theory. We will choose them to be zero, since the effects of these terms have already been analyzed elsewhere [9].

The renormalization scheme consists in a subtraction of the first three divergent terms of the De Witt-Schwinger-Christensen series from the expectation value (14):

$$\langle T_{\mu\nu}\rangle_{\rm ren} \ := \ \langle\Psi|T_{\mu\nu}|\Psi\rangle - \langle T_{\mu\nu}\rangle_{\rm DS\,div} \tag{15}$$

If $|\Psi\rangle$ is a state of finite energy density (compared with an adiabatic vacuum as will be explained later on), then the divergencies of $\langle\Psi|T_{\mu\nu}|\Psi\rangle$ and $\langle T_{\mu\nu}\rangle_{\rm DS\,div}$ will cancel and $\langle T_{\mu\nu}\rangle_{\rm ren}$ is finite. It should be noted, that the renormalization scheme decides about the physical meaning of the renormalized parameters.

With dimensional regularization ($d = 3 - \varepsilon$) one obtains [10]

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\rm DS\,div} \ &= \ \frac{1}{16\pi^2}\Big(\frac{1}{\varepsilon} - \frac{1}{2}\Big(\gamma + \ln\frac{m^2}{4\pi}\Big)\Big)\cdot\Big(\frac{-4\,m^4}{(d{+}1)(d{-}1)}\,g_{\mu\nu} - \frac{4\,m^2}{d-1}(\xi - \frac{1}{6})G_{\mu\nu} \\ &\qquad + \frac{1}{90}(H_{\mu\nu} - {}^{(2)}H_{\mu\nu}) + (\xi - \frac{1}{6})^2\,{}^{(1)}H_{\mu\nu}\Big),\end{aligned} \tag{16}$$

where γ is the Euler-Mascheroni constant. For our FRW metric (2) the H-tensors are explicitly given in appendix B.

2.1 de Sitter spacetime and Bunch-Davies vacuum

As already mentioned in the introduction we use the k=0 FRW parametrization of the de Sitter spacetime: $a(t) = e^{H_0 t}$ in (2). H_0 is the Hubble constant and due to the maximal symmetry we have $R_{\mu\nu} = -d\, H_0^2\, g_{\mu\nu}$ and $R = -d(d{+}1)\, H_0^2$.
By substituting the conformal time $\tau(t) := e^{-H_0 t}/H_0$ equation (10) becomes a Bessel differential equation:

$$\left((k\tau)^2\, \partial_{k\tau}^2 + k\tau\, \partial_{k\tau} + (k\tau)^2 - \nu^2\right)\, u(k,t) \;=\; 0 \tag{17}$$

where $\nu^2 := d^2/4 - (m^2{-}\xi R)/H_0^2$.
Its general solution is a linear combination of the two Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$:

$$u(k,t) \;=\; B_1(k)\, H_\nu^{(1)}(k\tau) + B_2(k)\, H_\nu^{(2)}(k\tau) \tag{18}$$

A comparison with the adiabatic vacuum in section 3.1 shows that we have to require $B_1(k) \stackrel{k\to\infty}{\to} 0$, since our quantum state should have a finite energy density. In addition we want the quantum state not to break the de Sitter symmetry. It follows that ρ and p have to be constant over the whole de Sitter manifold. This is the case if B_1 and B_2 in (18) are independent of k, as can be seen by substituting $y := k\tau$ in the integrals (14).
Therefore we end up with $B_1(k) = 0$ and $u(k,t) = H_\nu^{(2)}(k\tau(t))$, leading to

$$\frac{1}{2\,\mathrm{Re}A(k,t)} \;=\; \frac{\pi}{4}\, H_0^{d-1}\tau^d H_\nu^{(1)}(k\tau)\, H_\nu^{(2)}(k\tau). \tag{19}$$

The quantum state specified in this manner is known as the Bunch-Davies vacuum [11, 12].
Using (11), (12) and (19) it turns out that the integrals (14) involving two Bessel functions are of the Weber-Schafheitlin type and can be evaluated analytically. The results are given in appendix A. Expansion in $\varepsilon = 3{-}d$ yields ($\psi(x) = \Gamma'(x)/\Gamma(x)$)

$$\langle\Psi|T_{\mu\nu}|\Psi\rangle \stackrel{\varepsilon\to 0}{=} \frac{-g_{\mu\nu}}{d{+}1}\, \frac{m^2 H_0^2}{16\pi^2} \Big(\frac{1}{4} {-} \nu^2\Big) \Big(\frac{2}{\varepsilon} {-} \gamma {+} 1 {+} \ln\frac{4\pi}{H_0^2} {-} \psi(\tfrac{3}{2} {+} \nu) - \psi(\tfrac{3}{2} {-} \nu) {+} \mathcal{O}(\varepsilon)\Big). \tag{20}$$

The $1/\varepsilon$-pole can be removed by a renormalization through the subtraction of De Witt-Schwinger terms as described in the foregoing section, and we obtain the final result:

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\text{ren}} &= \frac{1}{64\pi^2}\, g_{\mu\nu}\Big(m^2(m^2-(\xi-\tfrac{1}{6})R)\Big(\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln\frac{H_0^2}{m^2}\Big) \\ &\qquad + m^2(\xi-\tfrac{1}{6})R+\tfrac{1}{18}\,m^2R-a_2\Big) \end{aligned} \tag{21}$$

with $a_2 = -\frac{1}{2160}\,R^2+\frac{1}{2}(\xi-\frac{1}{6})^2R^2$, $R=-12\,H_0^2$.

Due to the de Sitter symmetry this expectation value is proportional to the metric tensor and acts just like an effective cosmological constant within the Einstein equations. Therefore a de Sitter spacetime with its Hubble constant H_0 determined by the transzendental equation

$$3\,H_0^2+\Lambda \;=\; 8\pi G_N\,\rho_{\text{ren}}(H_0^2) \tag{22}$$

together with the Bunch-Davies vacuum forms a solution of the semiclassical, coupled system of equations (1).

The solutions of (22) for given Λ have been studied in ref. [13]. Clearly, for every given H_0 there is a Λ so that (22) is fulfilled. Thus a de Sitter spacetime of arbitrary curvature is possible.

3 Nearly de Sitter spacetimes

We are now approaching our main goal: the linear stability analysis of the semiclassical solution from the foregoing chapter. Our starting point is that of reference [5]: We will consider small fluctuations of the gravitational field and small perturbations of the matter quantum state. The semiclassical equations are linearized around the de Sitter – Bunch-Davies solution. At time t_0 the whole system will be given an initial configuration, which slightly deviates from de Sitter spacetime and Bunch-Davies vacuum. Then we analyse the time evolution of this deviation. Any instabilities would be indicated by (exponentially) growing parts in the general

solution for the deviation.

For simplicity and feasibility only fluctuations of the metric will be considered, which do not break its spatial homogeneity and isotropy. This is of course a limitation, but it has already been shown (for example in reference [14]) that small initial anisotropies are damped away by particle production and an automatic isotropization takes place.

The Gaussian form of the wave functional is not altered by the metric fluctuations. This is also assumed for its initial deviation. Due to the linearization this assumption does not exclude an initial wave functional containing first excitations. Moreover the energy density of initial excitations would be subject to the exponential de Sitter red-shift and have no influence on the long term behaviour of the system.

Firstly we want to compute the change in the Gaussian width of the wave functional and in the energy momentum components (14) for a given fluctuation of the FRW scale factor and initial deviation from the Bunch-Davies vacuum. In the sequel quantities related to the unperturbed de Sitter spacetime and Bunch-Davies vacuum will get the index 0, whereas a prefix δ always means the deviation of a quantity from its unperturbed value.

In order to save some ink the Hubble parameter H_0 of the unperturbed de Sitter spacetime will be set equal to 1 (in addition to $\hbar$ and c). This means that masses are measured in units of H_0.

Since the results do not depend on the starting time, $t_0 = 0$ will be used without loss of generality.

Consider now a small deviation of the FRW scale factor from its de Sitter value $a_0(t) = e^t$:

$$
\begin{aligned}
a(t) &= a_0(t)\,(1 + I(t))\,, \qquad I(t) \ll 1 \\
H(t) &= \frac{\dot{a}}{a} = 1 + \dot{I}(t)
\end{aligned}
\tag{23}
$$

In the following every quantity will be linearized with respect to $I(t)$. The components of the energy momentum tensor are written as

$$\begin{aligned}\rho(t) &= \langle T_{00}\rangle \;=\; \rho_0+\delta\rho(t)\\ p(t) &= -\frac{1}{d}\,g^{ij}\langle T_{ij}\rangle \;=\; p_0+\delta p(t)\;,\end{aligned}$$

where ρ_0 and p_0 are the unperturbed quantities (21). There are two sources of contributions to $\delta\rho$ und δp: The first one emerges from the explicit appearance of the metric in the definition of $T_{\mu\nu}$, and the other one is due to the dependence of the Hamiltonian on the metric leading to a deviation of the wave functional from the Bunch-Davies vacuum:

$$A(k,t) \;=\; A_0(k,t)+\delta A(k,t)$$

With the help of (11), (12) and by noting that $\partial_t\int\tilde{\tilde{dk}}/(2\,\mathrm{Re}A_0(k,t))=0$ we obtain from (14):

$$\begin{aligned}\delta\rho &= \xi\,(\delta R_{00}-\delta R)\int\tilde{\tilde{dk}}\,\frac{1}{2\,\mathrm{Re}A_0}\;+\int\tilde{\tilde{dk}}\,k^2\delta\Big(\frac{a^{-2}}{2\,\mathrm{Re}A}\Big)\\ &\quad+\Big(m^2+\xi\,(R_{00}-R)+d\,(\xi+\tfrac14)\,\partial_t+\tfrac14\,\partial_t^2\Big)\int\tilde{\tilde{dk}}\,\delta\Big(\frac{1}{2\,\mathrm{Re}A}\Big)\\ \delta p &= \xi\,\delta(a^{-2}R_{11})\int\tilde{\tilde{dk}}\,\frac{1}{2\,\mathrm{Re}A_0}\;+\frac{1}{d}\int\tilde{\tilde{dk}}\,k^2\delta\Big(\frac{a^{-2}}{2\,\mathrm{Re}A}\Big)\\ &\quad+\Big(\xi\,a^{-2}R_{11}+(\tfrac{d}{4}+\xi(1-d))\,\partial_t+(\tfrac14-\xi)\,\partial_t^2\Big)\int\tilde{\tilde{dk}}\,\delta\Big(\frac{1}{2\,\mathrm{Re}A}\Big)\end{aligned}\qquad(24)$$

Since $\mathrm{Re}A(k,t)=\sqrt{g(t)}\,\mathrm{Re}\,\Gamma(k,t)$ we have

$$\delta\Big(\frac{1}{2\,\mathrm{Re}A}\Big) \;=\; -\frac{d\,I}{2\,\mathrm{Re}A_0}-F_k\;,\qquad \delta\Big(\frac{a^{-2}}{2\,\mathrm{Re}A}\Big) \;=\; -a_0^{-2}\frac{(d+2)\,I}{2\,\mathrm{Re}A_0}-a_0^{-2}F_k\qquad(25)$$

with $F_k:=-a_0^{-d}\delta\Big(\dfrac{1}{2\,\mathrm{Re}\,\Gamma}\Big)$.

The Schrödinger equation (9) leads to the following equation of motion for $\delta\Gamma(k,t)$:

$$i\,\partial_t\delta\Gamma(k,t)-2\,\Gamma_0(k,t)\,\delta\Gamma(k,t) \;=\; 2\,I(t)\,a_0^{-2}k^2+r(t)\qquad(26)$$

with

$$\begin{aligned}r(t) &:= \frac{d^2}{2}\,\dot I(t)+\frac{d}{2}\,\ddot I(t)+\xi\,\delta R(t) \;=\; \frac12\,\dot I(t)+\frac12\,\ddot I(t)+\xi_c\delta R(t)\;,\\ \xi_c &:= \xi-\frac{d-1}{4d}\;.\end{aligned}\qquad(27)$$

From the foregoing section we know that $\Gamma_0(k,t) = -i\,\partial_t \ln H_\nu^{(2)}(k\tau_0(t))$, $\tau_0(t) = a_0^{-1} = e^{-t}$, so that the general solution of (26) is

$$\begin{aligned}
\delta\Gamma(k,t) \;&=\; -\,\frac{i}{{H_\nu^{(2)}}^2(k\tau_0(t))}\int_0^t dt'\,\Big(r(t') + 2\,I(t')\,a_0^{-2}(t')\,k^2\Big)\,{H_\nu^{(2)}}^2(k\tau_0(t'))\\
&\quad+\,\frac{{H_\nu^{(2)}}^2(k)}{{H_\nu^{(2)}}^2(k\tau_0(t))}\,\delta\Gamma(k,0) \qquad (28)
\end{aligned}$$

Using this and (19) we find

$$\begin{aligned}
F_k \;&=\; a_0^{-d}\frac{\mathrm{Re}\,\delta\Gamma}{2\,(\mathrm{Re}\,\Gamma_0)^2} \;=\; F_k^{(i)} + F_k^{(ii)}\\
F_k^{(i)} \;&:=\; a_0^{-d}\,\mathrm{Re}\Big(-i\,\frac{\pi^2}{8}\,{H_\nu^{(1)}}^2(k\tau_0(t))\int_0^t dt'\,\Big(r(t') + 2\,I(t')\,a_0^{-2}(t')\,k^2\Big)\,{H_\nu^{(2)}}^2(k\tau_0(t'))\Big)\\
F_k^{(ii)} \;&:=\; a_0^{-d}\,\mathrm{Re}\Big(\frac{\pi^2}{8}\,{H_\nu^{(1)}}^2(k\tau_0(t))\,{H_\nu^{(2)}}^2(k)\,\delta\Gamma(k,0)\Big)\,. \qquad (29)
\end{aligned}$$

The initial deviation $\delta\Gamma(k,0)$ from the Bunch-Davies vacuum $\Gamma_0(k,0)$ is part of the initial data of the problem. Since a physically meaningful, perturbed initial state should have a finite energy density, $\delta\Gamma(k,0)$ must have a special high energy behaviour:

$$\delta\Gamma(k,0) \;=\; \delta\Gamma^{(ii)}(k) - i\sum_{n=-3}^{+1}(ik)^n\delta\Gamma_n\,, \qquad (30)$$

where $\delta\Gamma^{(ii)}(k) \overset{k\to\infty}{\longrightarrow} 0$ faster than k^{-3}, and the coefficients $\delta\Gamma_n$ are determined by a comparison with the adiabatic vacuum in the next section. Of course $\delta\Gamma(k,0)$ has to be finite for $k \to 0$. This can be ensured by suitable $\delta\Gamma^{(ii)}(k)$ in the form (30).

3.1 The adiabatic vacuum

Although it is not possible to define a unique vacuum state in a general curved background spacetime, it is possible to define a state which is vacuos for the high k-modes in the limit $k \to \infty$. This can be achieved by using the adiabatic expansion of positive frequency for the field modes, which is at the same time an expansion in k^{-1} and becomes exact in the above limit. We require the width $\Gamma(k,0)$ of our

quantum state to coincide with the width $\Gamma_{\rm ad}(k,0)$ of the adiabatic vacuum in the limit of large k, so our energy momentum tensor expectation value will have the same divergencies as the De Witt-Schwinger one (being also a local expansion). This yields a finite renormalized energy momentum density.

Since the energy momentum expectation value is quartically divergent, the terms up to a relative order of k^{-4} in the Gaussian width $\Gamma_{\rm ad}(k,t) \sim k$ are responsible for its divergencies. "In the limit of large k" means therefore "up to the relative order of k^{-4} in the limit $k \to \infty$".

We are now proceeding with the computation of $\Gamma_{\rm ad}(k,t)$ by an adiabatic expansion (positive frequency) of the solution of (10). Again the conformal time $\tau := -\int_{t_0}^{t} dt'/a(t')$ with $dt^2 = a^2(t)\, d\tau^2$ is introduced. We substitute $u_{\rm ad}(k,t) =: a^{1/2}(t)\, \chi_k(\tau)$ and (10) takes the form

$$\partial_\tau^2\, \chi_k(\tau) + \Omega_k^2(\tau)\, \chi_k(\tau) \;=\; 0 \tag{31}$$

with $\Omega_k(\tau) := (k^2 + a^2 \tilde{m}^2)^{1/2}\,, \qquad \tilde{m}^2 := m^2 - \xi_c R\,.$

The adiabatic solutions of positive frequency are

$$\chi_k(\tau) \;=\; \frac{1}{\sqrt{2\, W_k(\tau)}}\, \exp\Big(-i \int_{\bar{\tau}}^{\tau} d\tau'\, W_k(\tau')\Big) \quad , \tag{32}$$

where the W_k have to obey the following equations:

$$W_k^2 + \frac{1}{2}\,\frac{W_k''}{W_k} - \frac{3}{4}\,\frac{W_k'^2}{W_k^2} \;=\; \Omega_k^2 \tag{33}$$

These are solved iteratively order by order ($' = d/d\tau$):

$$\begin{aligned}
W_k &= W_k^{(0)} + W_k^{(2)} + W_k^{(4)} + \;\cdots \\
W_k^{(0)} &= \Omega_k \\
W_k^{(2)} &= \frac{3}{8}\,\frac{\Omega_k'^2}{\Omega_k^3} - \frac{1}{4}\,\frac{\Omega_k''}{\Omega_k^2} \\
W_k^{(4)} &= \frac{1}{16}\,\frac{\Omega_k^{(\rm iv)}}{\Omega_k^4} - \frac{5}{8}\,\frac{\Omega_k'''\Omega_k'}{\Omega_k^5} - \frac{13}{32}\,\frac{\Omega_k''^2}{\Omega_k^5} + \frac{99}{32}\,\frac{\Omega_k''\Omega_k'^2}{\Omega_k^6} - \frac{297}{128}\,\frac{\Omega_k'^4}{\Omega_k^7}
\end{aligned}$$

Using the above Ω_k and expanding with respect to k^{-1} up to the fourth order relative to the leading one we obtain:

$$W_k \;=\; k + \frac{a^2}{k}\,\frac{\tilde{m}^2}{2} - \frac{a^4}{k^3}\,\frac{1}{8}\,(\tilde{m}^4 + 2\tilde{m}^2(\dot{H}+3H^2) - \xi_c(5H\dot{R}+\ddot{R})) + \mathcal{O}(k^{-4}) \quad (34)$$

$W_k^{(4)}$ and the first term in $W_k^{(2)}$ are already of the relative order of k^{-5} and do not appear in (34). Using $dt/d\tau = -a(t)$ the derivatives with respect to τ have been converted to those with respect to t. Putting things together we get the adiabatic width:

$$\begin{aligned}
\Gamma_{\mathrm{ad}}(k,t) \;&=\; -i\,\frac{\dot{u}_{\mathrm{ad}}(k,t)}{u_{\mathrm{ad}}(k,t)} \;=\; \frac{1}{a}\,W_k - i\,\Big(\frac{H}{2} - \frac{1}{2}\,\frac{\dot{W}_k}{W_k}\Big) \\
&=\; a^{-1}k - i\,\frac{H}{2} + \frac{1}{a^{-1}k}\,\frac{\tilde{m}^2}{2} + \frac{1}{a^{-2}k^2}\,\frac{i}{4}\,(2\,H\,\tilde{m}^2 - \xi_c\dot{R}) \\
&\quad - \frac{1}{a^{-3}k^3}\,\frac{1}{8}\,(\tilde{m}^4 + 2\tilde{m}^2(\dot{H}+3H^2) - \xi_c(5H\dot{R}+\ddot{R})) + \mathcal{O}(k^{-4}) \qquad (35)
\end{aligned}$$

This has to be compared with our Bunch-Davies width $\Gamma_0(k,t)$.

$$\Gamma_0(k,t) \;=\; -i\,\frac{\dot{u}_0(k,t)}{u_0(k,t)} \;=\; -i\,\frac{\partial_t H^{(2)}_{\nu_o}(k\tau_0)}{H^{(2)}_{\nu_o}(k\tau_0)} \;=\; i a_0^{-1} k\,\frac{\partial_{k\tau_o} H^{(2)}_{\nu_o}(k\tau_0)}{H^{(2)}_{\nu_o}(k\tau_0)}$$

Using the asymptotic expansion of the Hankel functions

$$\frac{H^{(2)\prime}_{\nu}(z)}{H^{(2)}_{\nu}(z)} \overset{|z|\to\infty}{=} -i - \frac{1}{2z} - i\,\frac{\frac{1}{4}-\nu^2}{2z^2} + \frac{\frac{1}{4}-\nu^2}{2z^3} + i\,\frac{(\frac{1}{4}-\nu^2)(\frac{25}{4}-\nu^2)}{8z^4} + \mathcal{O}(z^{-5})$$

as well as $\frac{1}{4} - \nu_0^2 = \tilde{m}_0^2/H_0^2$ we obtain:

$$\Gamma_0(k,t) = a^{-1}k - \frac{i}{2}H_0 + \frac{1}{a_0^{-1}k}\frac{\tilde{m}_0^2}{2} + \frac{i}{a_0^{-2}k^2}\frac{H_0\tilde{m}_0^2}{2} - \frac{1}{a_0^{-3}k^3}\frac{\tilde{m}_0^2}{8}(\tilde{m}_0^2 + 6H_0^2) + \mathcal{O}(k^{-4}) \quad (36)$$

For the sake of clarity we did not replace H_0 by 1 in this formula. The asymptotic expansion (36) coincides with (35) up to the order given in the special case of de Sitter spacetime ($\dot{H}_0 = \dot{R}_0 = \ddot{R}_0 = 0$): $\Gamma_0 = \Gamma_{\mathrm{ad}\,0} + \mathcal{O}(k^{-4})$. This means that $B_1(k \to \infty) \to 0$ was the correct choice in section 2.1.

For the nearly de Sitter spacetime we have to require $\delta\Gamma(k,0)+\Gamma_0(k,0) = \Gamma_{\mathrm{ad}}(k,0)+\mathcal{O}(k^{-4})$. After linearizing (35) with respect to the deviation from the de Sitter

spacetime we are finally in the position to obtain the coefficients $\delta\Gamma_n$ needed in (30):

$$\begin{aligned}
\delta\Gamma_1 &= -I(0)\,, \quad \delta\Gamma_0 \;=\; \frac{1}{2}\,\dot{I}(0)\,, \quad \delta\Gamma_{-1} \;=\; -\frac{1}{2}\,(\tilde{m}_0^2 I(0) - \xi_c \delta R(0))\,, \\
\delta\Gamma_{-2} &= \frac{1}{4}\,(4\tilde{m}_0^2 I(0) + 2\tilde{m}_0^2 \dot{I}(0) - \xi_c(2\delta R(0) + \delta\dot{R}(0)))\,, \\
\delta\Gamma_{-3} &= -\frac{1}{8}\,(3I(0)(\tilde{m}_0^4 + 6\tilde{m}_0^2) + 2\tilde{m}_0^2(\ddot{I}(0) + 6\dot{I}(0) - \xi_c \delta R(0)) \\
&\qquad - \xi_c(6\delta R(0) + 5\delta\dot{R}(0) + \delta\ddot{R}(0)))\,. \qquad (37)
\end{aligned}$$

3.2 Momentum integrals and isolation of divergencies

If we insert (25) into equation (24) there are two integrals of the Weber-Schafheitlin type involving two Hankel functions, which already appeared in section 2.1. After substituting $y := k\tau$ they are evaluated using (A.1):

$$\begin{aligned}
J_0^{(2)} &:= \int \tilde{\tilde{dk}}\, \frac{1}{2\,\mathrm{Re}A_0(k,t)} \\
&= -\frac{\tilde{m}_0^2}{16\pi^2}\Big(\frac{2}{\varepsilon} - \gamma + 1 + \ln 4\pi - \psi(\tfrac{3}{2}+\nu_0) - \psi(\tfrac{3}{2}-\nu_0) + \mathcal{O}(\varepsilon)\Big) \\
J_2^{(2)} &:= \int \tilde{\tilde{dk}}\, \frac{a_0^{-2}(t)\,k^2}{2\,\mathrm{Re}A_0(k,t)} \qquad (38) \\
&= \frac{\tilde{m}_0^2}{16\pi^2}\,\frac{3}{4}\,(m^2 - \xi R_0)\Big(\frac{2}{\varepsilon} - \gamma + \frac{5}{6} + \ln 4\pi - \psi(\tfrac{3}{2}+\nu_0) - \psi(\tfrac{3}{2}-\nu_0) + \mathcal{O}(\varepsilon)\Big)
\end{aligned}$$

The other integrals appearing in (24) are involving four Hankel functions. In terms of

$$\begin{aligned}
J_l^{(4)}(t-t') &:= a_0^{-d}(t)\int \tilde{\tilde{dk}}\,(k\tau_0(t'))^l\, H_\nu^{(1)^2}(k\tau_0(t))\, H_\nu^{(2)^2}(k\tau_0(t')) \\
&= \frac{2^{1-d}\tau_0^d(t-t')}{\Gamma(\frac{d}{2})\,\pi^{d/2}} \int_0^\infty dy\, y^{l+d-1}\, H_\nu^{(1)^2}(y\tau_0(t-t'))\, H_\nu^{(2)^2}(y) \qquad (39)
\end{aligned}$$

they explicitly read

$$\int \tilde{\tilde{dk}}\, F_k^{(i)} \;=\; \mathrm{Re}\Big(-i\,\frac{\pi^2}{8}\int_0^t dt'\,\Big(r(t')\,J_0^{(4)}(t-t') + 2I(t')\,J_2^{(4)}(t-t')\Big)\Big) \qquad (40)$$

$$
\begin{aligned}
\int \tilde{d\tilde{k}}\, a_0^{-2} k^2 F_k^{(i)} &= \mathrm{Re}\Big(-i\,\frac{\pi^2}{8}\int_0^t dt'\,\tau_0^2(t-t')\,\Big(r(t')\,J_2^{(4)}(t-t') + 2I(t')\,J_4^{(4)}(t-t')\Big)\Big) \\
\int \tilde{d\tilde{k}}\, F_k^{(ii)} &= \mathrm{Re}\Big(-i\,\frac{\pi^2}{8}\sum_{n=-3}^{1}\delta\Gamma_n\, i^n J_n^{(4)}(t)\Big) + \delta\Gamma^{(ii)}\text{-terms} \qquad (41) \\
\int \tilde{d\tilde{k}}\, a_0^{-2} k^2 F_k^{(ii)} &= \mathrm{Re}\Big(-i\,\frac{\pi^2}{8}\,\tau_0^2(t)\sum_{n=-3}^{1}\delta\Gamma_n\, i^n J_{n+2}^{(4)}(t)\Big) + \delta\Gamma^{(ii)}\text{-terms}\,.
\end{aligned}
$$

The t'-integrations in (40) are convolution integrals. The fact that the $J_l^{(4)}$ defined above depend only on the difference $t-t'$ means independence of the starting time ($t_0 = 0$ here) and follows from the maximum symmetry of de Sitter spacetime.

The integrals (39) involving a product of four Hankel resp. Bessel functions were not found in the mathematical standard literature. Therefore their evaluation has been included as part of this work in appendix A.

At this point the investigations in references [5] and [6] failed. In order to circumvent the integrations of four Hankel functions they carry out a so called "short-time" approximation, which consists in a restriction on short times t and the Taylor series expansion of $J_l^{(4)}(t-t')$ around $t' = t$ up to the linear order. However, the $J_l^{(4)}(t-t')$ have a singularity at $t' = t$ (see below) and cannot be expanded around this point. Hence this approximation does not lead to correct results even for arbitrarily short times. Moreover the divergencies are not coming out correctly, so no sensible renormalization is possible. Comparing with our exact results (65) and (66) it turns out that even the leading terms of a real short-time approximation are missing.

In reference [7] the momentum integrations are not executed and one is left with even more complicated integrals in the final result.

We show in appendix A Eq. (A.10) that

$$
J_l^{(4)}(t) = -\frac{2^{3-d}(-i)^{d+l}}{\Gamma(\frac{d}{2})\,\pi^{2+d/2}}\; G(t;0,l)\,. \qquad (42)
$$

The dependence of G and $J_l^{(4)}$ on d and ν has been suppressed in favour of a shorter notation. The definition of G in appendix A explictly reads:

$$
\begin{aligned}
G(t;p,l) \;:=\; & \frac{-\tau_0^d(t)}{4\sin^2\pi\nu}\Big(\\
& -\, 2\; {}_4\bar{F}_3\Big(\frac{d+l}{2}-p, \frac{d+l}{2}-\nu, \frac{d+l}{2}+\nu, \frac{1}{2}; 1+\nu, 1-\nu, \frac{d+l+1}{2}; \tau_0^2(t)\Big) \\
& +\tau_0^{-2\nu}(t)\, e^{2\pi i\nu}\, {}_4\bar{F}_3\Big(\frac{d+l}{2}-\nu-p, \frac{d+l}{2}-2\nu, \frac{d+l}{2}, \frac{1}{2}-\nu; 1-\nu, 1-2\nu, \frac{d+l+1}{2}-\nu; \tau_0^2(t)\Big) \\
& +\tau_0^{2\nu}(t)\, e^{-2\pi i\nu}\, {}_4\bar{F}_3\Big(\frac{d+l}{2}+\nu-p, \frac{d+l}{2}, \frac{d+l}{2}+2\nu, \frac{1}{2}+\nu; 1+\nu, 1+2\nu, \frac{d+l+1}{2}+\nu; \tau_0^2(t)\Big)\Big)
\end{aligned}
\tag{43}
$$

Our generalized hypergeometric function ${}_4\bar{F}_3$ appearing in (43) is defined by an infinite series:

$$
\begin{aligned}
& {}_4\bar{F}_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z) \;:= \\
& \qquad \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(\alpha_4)}{\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}\; {}_4F_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z) \\
& \quad = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)\,\Gamma(\alpha_3+n)\,\Gamma(\alpha_4+n)}{n!\;\Gamma(\beta_1+n)\,\Gamma(\beta_2+n)\,\Gamma(\beta_3+n)}\; z^n \,,
\end{aligned}
\tag{44}
$$

where ${}_pF_q$ is the function usually called generalized hypergeometric function in the mathematical literature.

According to appendix A the integral (39) is convergent for $\tau_0(t-t') \neq 1$ resp. $t' \neq t$ in the region $4\,|\mathrm{Re}\,\nu| < l + \mathrm{Re}\,d < 3$ (which has always a non-zero extension if $m^2–\xi R_0 > 0$). Equation (42) gives its analytic continuation on the whole complex d-plane.

In order to investigate the convergence behaviour of the series (44), we need an asymptotic expansion of its terms for large n. This can be obtained using Stirling's series for the gamma-function (see [15, §13.6]):

$$
\Gamma(\alpha+n) \;\stackrel{n\to\infty}{=}\; \sqrt{2\pi}\, e^{-n} n^{n+\alpha-\frac{1}{2}} \exp\Big(\sum_{m=1}^{M} \frac{(-)^{m+1} B_{m+1}(\alpha)}{m(m+1)\,n^m} + \mathcal{O}\Big(\frac{1}{n^{M+1/2}}\Big)\Big), \tag{45}
$$

where $B_m(\alpha)$ are the Bernoulli polynoms:

$$
\begin{aligned}
& B_0(x) = 1,\; B_1(x) = x - \frac{1}{2}\,,\; B_2(x) = x^2 - x + \frac{1}{6}\,,\; B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}\,, \\
& B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}\,,\; B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x\,,\; \ldots
\end{aligned}
\tag{46}
$$

Defining

$$\bar{B}_m(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3,\beta_4) \;:=\; \sum_{i=1}^4 B_m(\alpha_i) - \sum_{i=1}^4 B_m(\beta_i)\,,$$
$$\sigma \;:=\; \sum_{i=1}^4 \alpha_i - \sum_{i=1}^3 \beta_i - 1 \tag{47}$$

we get

$$\begin{aligned} Q_n \;&:=\; \frac{\Gamma(\alpha_1{+}n)\,\Gamma(\alpha_2{+}n)\,\Gamma(\alpha_3{+}n)\,\Gamma(\alpha_4{+}n)}{\Gamma(\beta_1{+}n)\,\Gamma(\beta_2{+}n)\,\Gamma(\beta_3{+}n)\,\Gamma(1{+}n)} \\ &\overset{n\to\infty}{=}\; n^\sigma \exp\Big(\sum_{m=1}^M \frac{(-)^{m+1}\bar{B}_{m+1}(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3,1)}{m(m+1)\,n^m} + \mathcal{O}\Big(\frac{1}{n^{M+1/2}}\Big)\Big) \\ &\overset{M=4}{=}\; n^\sigma\Big(1+\frac{\tilde{B}_1(\ldots)}{n}+\frac{\tilde{B}_2(\ldots)}{n^2}+\frac{\tilde{B}_3(\ldots)}{n^3}+\frac{\tilde{B}_4(\ldots)}{n^4}+\mathcal{O}\Big(\frac{1}{n^{9/2}}\Big)\Big)\,. \end{aligned} \tag{48}$$

The polynoms $\tilde{B}_m$ arise from an expansion of the exponential function and are polynoms of Bernoulli polynoms:

$$\begin{aligned} &\tilde{B}_1(\ldots) := \frac{\bar{B}_2(\ldots)}{2}\,,\ \tilde{B}_2 := \frac{\bar{B}_2^2}{8}-\frac{\bar{B}_3}{6}\,,\ \tilde{B}_3 := \frac{\bar{B}_2^3}{48}-\frac{\bar{B}_2\bar{B}_3}{12}+\frac{\bar{B}_4}{12} \\ &\tilde{B}_4 := \frac{\bar{B}_2^4}{384}-\frac{\bar{B}_2^2\bar{B}_3}{48}+\frac{\bar{B}_2\bar{B}_4}{24}+\frac{\bar{B}_3^2}{72}-\frac{\bar{B}_5}{20}\,,\ \tilde{B}_0 := 1\,. \end{aligned} \tag{49}$$

The series (44) is always convergent for $0 \le z < 1$ respectively $t > 0$ in (43). Putting in the arguments $\alpha_1 \ldots \beta_3$ from (43) we find $\sigma = d+l-3-p$ and therefore $0 \le \sigma \le 4$ for $d = 3$, $l = 0,2,4$ and $p = 0$. This means that in the limit $z \to 1$ ($t \to 0$ in (43)) the leading terms of the series are the ones of large n, which can be computed using the expansion (48).

Introducing the function

$$F(z,s) \;=\; \sum_{n=1}^\infty \frac{z^n}{n^s} \tag{50}$$

we obtain

$$\begin{aligned} {}_4\bar{F}_3(\alpha_1,\ldots,\alpha_4;\beta_1,\ldots,\beta_3;z) \;&=\; \sum_n Q_n z^n \\ &\overset{z\to1}{=}\; F(z,-\sigma)+\tilde{B}_1\,F(z,-\sigma{+}1)+\tilde{B}_2\,F(z,-\sigma{+}2) \\ &\quad+\tilde{B}_3\,F(z,-\sigma{+}3)+\tilde{B}_4\,F(z,-\sigma{+}4)+\ \ldots\ . \end{aligned} \tag{51}$$

According to [16, chapter 1.11] the function F obeys the following relations ($B_m = B_m(0)$ Bernoulli numbers):

$$\begin{aligned}
F(z,-m) &= m!\,(-\ln z)^{-m-1} - \sum_{r=0}^{\infty} \frac{B_{m+r+1}}{(m+r+1)\,r!}\,(\ln z)^r\,, \qquad m = 1,2,3,\ldots \\
F(z,0) &= \frac{z}{1-z}\,, \qquad F(z,1) \;=\; \ln(1-z) \\
F(1,s) &= \zeta(s) \quad \text{Riemann's zeta-function}\,.
\end{aligned} \tag{52}$$

Combining the results obtained so far it follows that

$$J_l^{(4)}(t-t') \overset{t' \to t}{\sim} (t-t')^{-(d+l-2)}\,. \tag{53}$$

This means that due to the behaviour of the integrand at the upper limit of integration the convolutions in (40) are logarithmically ($d = 3$, $l = 0$), quadratically ($d = 3$, $l = 2$) or quartically ($d = 3$, $l = 4$) divergent, if we suppose $r(t')$ and $I(t')$ to be smooth functions different from zero.

The divergent convolutions in (40) represent non-local contributions to the energy momentum expectation value (24). However the divergencies emerge directly at the upper limit of integration and involve the functions r and I (respectively the spacetime metric) only at the point $t' = t$. Hence the divergencies are again of a local nature as claimed in section 2.

For the purpose of renormalization we have to isolate the divergencies from the integrals (40) in terms of $1/\varepsilon$-poles ($\varepsilon = 3-d$). Taking into account the behaviour (53) of the convolution kernels it turns out that this can be achieved by performing $l+1$ integrations by parts. One obtains divergent as well as finite boundary terms and finite convolution integrals. At this point the variable p appearing already in (43) (being unused up to now) becomes meaningful: According to appendix A our G-function satisfies the relations

$$\begin{aligned}
\Big(\frac{1}{2}\,\partial_{t'} - \Big(p+1-\frac{l}{2}\Big)\Big) G(t-t';p+1,l) &= G(t-t';p,l) \qquad (54) \\
\Big(\frac{1}{2}\,\partial_{t'} - \Big(p+2-\frac{l}{2}\Big)\Big) \tau_0^2(t-t')\, G(t-t';p+1,l) &= \tau_0^2(t-t')\, G(t-t';p,l)\,,
\end{aligned}$$

which can be used to do the integrations by parts in (40). The variable p then stands for the number of integrations by parts performed so far. In this way we obtain for the integrals appearing in (40):

$$
\begin{aligned}
&\int_0^t dt'\, r(t')\, G(t-t';0,0) \;=\\
&\qquad \frac{1}{2}\, r(t')\, G(t-t';1,0)\Big|_{t'=0}^{t'=t} - \int_0^t dt' \Big(\frac{1}{2}\,\dot r(t') + r(t')\Big)\, G(t-t';1,0)\\
&\int_0^t dt'\, I(t')\, G(t-t';0,2) \;=\\
&\qquad \frac{1}{2}\Big(I(t')\, G(t-t';1,2) - \frac{\dot I(t')}{2}\, G(t-t';2,2) + \Big(\frac{\ddot I(t')}{4} + \frac{\dot I(t')}{2}\Big) G(t-t';3,2)\Big)\Big|_0^t\\
&\qquad - \int_0^t dt' \Big(\frac{\dddot I(t')}{8} + \frac{3}{4}\,\ddot I(t') + \dot I(t')\Big)\, G(t-t';3,2)\\
&\int_0^t dt'\, r(t')\, \tau_0^2(t-t') G(t-t';0,2) \;=\\
&\qquad \frac{1}{2}\Big(r(t')\, \tau_0^2 G(t-t';1,2) - \Big(\frac{\dot r(t')}{2} + r(t')\Big)\, \tau_0^2 G(t-t';2,2)\\
&\qquad\qquad + \Big(\frac{\ddot r(t')}{4} + \frac{3}{2}\,\dot r(t') + 2\, r(t')\Big)\, \tau_0^2 G(t-t';3,2)\Big)\Big|_0^t\\
&\qquad - \int_0^t dt' \Big(\frac{\dddot r(t')}{8} + \frac{3}{2}\,\ddot r(t') + \frac{11}{2}\,\dot r(t') + 6\, r(t')\Big)\, \tau_0^2 G(t-t';3,2)\\
&\int_0^t dt'\, I(t')\, \tau_0^2(t-t') G(t-t';0,4) \;=\\
&\qquad \frac{1}{2}\Big(I(t')\, \tau_0^2 G(t-t';1,4) - \frac{\dot I(t')}{2}\, \tau_0^2 G(t-t';2,4)\\
&\qquad\quad + \Big(\frac{\ddot I(t')}{4} + \frac{\dot I(t')}{2}\Big)\, \tau_0^2 G(t-t';3,4) - \Big(\frac{\dddot I(t')}{8} + \frac{3}{4}\,\ddot I(t') + \dot I(t')\Big)\, \tau_0^2 G(t-t';4,4)\\
&\qquad\qquad + \Big(\frac{I^{(\mathrm{iv})}(t')}{16} + \frac{3}{4}\,\dddot I(t') + \frac{11}{4}\,\ddot I(t') + 3\,\dot I(t')\Big)\, \tau_0^2 G(t-t';5,4)\Big)\Big|_0^t\\
&\qquad - \int_0^t dt' \Big(\frac{I^{(\mathrm{v})}(t')}{32} + \frac{5}{8}\, I^{(\mathrm{iv})}(t') + \frac{35}{8}\,\dddot I(t') + \frac{25}{2}\,\ddot I(t') + 12\,\dot I(t')\Big)\, \tau_0^2 G(t-t';5,4)
\end{aligned} \tag{55}
$$

The convolutions we are left with in (55) are convergent and finite. The same holds for the lower boundary terms (proportional to $G(t;p,l)$) at $t>0$. The upper boundary terms proportional to $G(0;p,l)$ are divergent of the order $\sigma+1=d+l-2-p$, because they contain the functions

$$\begin{aligned}
{}_4\bar{F}_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;1) \;&=\; \sum_{n=0}^{\infty} Q_n \;=\; \sum_n n^\sigma + \ldots \\
=\;\; & \zeta(-\sigma) + \tilde{B}_1\,\zeta(-\sigma{+}1) + \tilde{B}_2\,\zeta(-\sigma{+}2) + \tilde{B}_3\,\zeta(-\sigma{+}3) + \tilde{B}_4\,\zeta(-\sigma{+}4) \\
& + \text{finite terms}\;. \qquad (56)
\end{aligned}$$

In the second line above we have already given the analytic continuation of the divergent part of the series ${}_4\bar{F}_3(\ldots;\ldots;1)$ using Riemann's zeta-function. We have taken into account as many terms as necessary for the "most divergent" case $G(0;1,4)$ with $\sigma=d=3-\varepsilon$.

The zeta-function has exactly one simple pole at $s=1$:

$$\zeta(s) \;=\; \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n\;, \qquad (57)$$

where γ is the Euler-Mascheroni constant (see [16]). This pole becomes an $1/\varepsilon$-pole in (56) and represents the divergencies in the typical manner for dimensional regularization. In order to separate them from the finite part we rewrite (56) in the form

$${}_4\bar{F}_3(\alpha_1,\ldots,\beta_3;1) \;=\; {}_4\tilde{F}_3(\alpha_1,\ldots,\beta_3) + \sum_{m=0}^{M} \tilde{B}_m(\alpha_1,\ldots,\beta_3,1)\,\zeta(m-\sigma)\,, \qquad (58)$$

where we have defined the convergent series ${}_4\tilde{F}_3$ in the following way:

$$\begin{aligned}
& {}_4\tilde{F}_3(\alpha_1,\ldots,\beta_3) \;:= \qquad (59) \\
& \qquad Q_0(\alpha_1,\ldots,\beta_3) + \sum_{n=1}^{\infty}\Big(Q_n(\alpha_1,\ldots,\beta_3) - \sum_{m=0}^{M} \tilde{B}_m(\alpha_1,\ldots,\beta_3,1)\, n^{\sigma-m}\Big)
\end{aligned}$$

The number $M+1$ of terms to be subtracted from every term of the series is determined by $\alpha_1\ldots\beta_3$ in such a way that the series (59) is just convergent: M is the biggest integer less or equal to $\sigma+1$, and in our case for $d=3$ we have

$M = l - p + 1$.

In (59) $d = 3$ may already be substituted. The terms of the series behave like $1/n^2$ for large n, hence a truncation of the series at $n = N$ (for example for an approximate numerical calculation) will lead to an error of the order of $1/N$.

The only term in (58), for which the regularization $d = 3 - \varepsilon$ has to be retained until renormalization, is the $m = M$-term in the zeta-function sum containing the $1/\varepsilon$-pole.

Now we define a finite function $\tilde{G}(p, l)$ in the same manner as previously $G(t; p, l)$, excepted that $t = 0$ and the ${}_4\bar{F}_3$'s in the definition (43) have to be replaced by the corresponding ${}_4\tilde{F}_3$'s.

The functions $G(0; p, l)$ which are needed for the upper boundary terms in (55) can then be expressed in terms of $\tilde{G}(p, l)$ and zeta-functions. Using (43), (58), (49), (47) und (46) we obtain in the limit $\varepsilon \to 0$:

$$
\begin{aligned}
G(0;1,0) &= \tilde{G}(1,0) + \zeta(1+\varepsilon) \\
G(0;1,2) &= \tilde{G}(1,2) + \zeta(-1) + (1 + i\nu \cot \pi\nu)\,\zeta(0) \\
&\quad + \Big(\frac{3}{2}\Big(\frac{1}{4} - \nu^2\Big) + \varepsilon\Big(\frac{\nu^2}{4\sin^2 \pi\nu} - \frac{29}{48} - i\nu \cot \pi\nu\Big)\Big)\zeta(1+\varepsilon) \\
G(0;2,2) &= \tilde{G}(2,2) + \zeta(0) + \Big(\frac{1}{2} - \varepsilon\Big(\frac{3}{4} + i\nu \cot \pi\nu\Big)\Big)\zeta(1+\varepsilon) \\
G(0;3,2) &= \tilde{G}(3,2) + \zeta(1+\varepsilon) \\
G(0;1,4) &= \tilde{G}(1,4) + \zeta(-3) + \Big(\frac{9}{2} + 3i\nu \cot \pi\nu\Big)\zeta(-2) \\
&\quad + \Big(\frac{57}{8} + \frac{\nu^2}{2} - \frac{3\nu^2}{2\sin^2 \pi\nu} + 9i\nu \cot \pi\nu\Big)\zeta(-1) \\
&\quad + \Big(\frac{29}{8} + 2\nu^2 - \frac{9\nu^2}{4\sin^2 \pi\nu} + i\Big(\frac{57}{8}\nu - \frac{3}{2}\nu^3\Big) \cot \pi\nu\Big)\zeta(0) \\
&\quad + \Big(\frac{15}{4}\Big(\frac{1}{4} - \nu^2\Big) + \frac{15}{8}\Big(\frac{1}{4} - \nu^2\Big)^2 \\
&\qquad - \varepsilon\Big(\frac{407}{240} + \frac{47}{96}\nu^2 + \frac{\nu^4 - 57\nu^2/16}{2\sin^2 \pi\nu} - i\Big(\nu^3 - \frac{29}{8}\nu\Big) \cot \pi\nu\Big)\Big)\zeta(1+\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
G(0;2,4) \;&=\; \tilde{G}(2,4)+\zeta(-2)+(3+2i\nu\cot\pi\nu)\,\zeta(-1) \\
&\quad+\Big(\frac{21}{8}-\frac{3}{2}\nu^2-\frac{\nu^2}{2\sin^2\pi\nu}+3i\nu\cot\pi\nu\Big)\zeta(0)+\Big(-\frac{5}{4}\Big(\frac{1}{4}-\nu^2\Big) \\
&\qquad-\varepsilon\Big(\frac{49}{32}-\frac{11}{8}\nu^2-\frac{3\nu^2}{4\sin^2\pi\nu}+i\Big(\frac{21}{8}\nu-\frac{13}{6}\nu^3\Big)\cot\pi\nu\Big)\Big)\zeta(1+\varepsilon) \\
G(0;3,4) \;&=\; \tilde{G}(3,4)+\zeta(-1)+\Big(\frac{5}{2}+i\nu\cot\pi\nu\Big)\zeta(0) \\
&\quad+\Big(\frac{11}{8}-\frac{5}{2}\nu^2-\varepsilon\Big(\frac{83}{48}-\frac{\nu^2}{4\sin^2\pi\nu}+\frac{5}{2}\,i\nu\cot\pi\nu\Big)\Big)\zeta(1+\varepsilon) \\
G(0;4,4) \;&=\; \tilde{G}(4,4)+\zeta(0)+\Big(3-\varepsilon\Big(\frac{3}{4}+i\nu\cot\pi\nu\Big)\Big)\zeta(1+\varepsilon) \\
G(0;5,4) \;&=\; \tilde{G}(5,4)+\zeta(1+\varepsilon)
\end{aligned}
\tag{60}
$$

The zeta-functions have the following values

$$
\begin{aligned}
&\zeta(1+\varepsilon) \;=\; \frac{1}{\varepsilon}+\gamma\,, \quad \zeta(-n) \;=\; \frac{(-)^n B_{n+1}}{n+1} \quad n\in\mathbb{N}_0 \\
\Rightarrow\quad &\zeta(0) \;=\; -\frac{1}{2}\,, \quad \zeta(-1) \;=\; -\frac{1}{12}\,, \quad \zeta(-2) \;=\; 0\,, \quad \zeta(-3) \;=\; \frac{1}{120}\,.
\end{aligned}
\tag{61}
$$

With the help of (25)–(30), (38), (40), (41), (42), (55), (60) and (61) we are in the position to specify the rest of the integrals appearing in (24):

$$
\begin{aligned}
\int \tilde{d\tilde{k}}\,\delta\Big(\frac{1}{2\,\mathrm{Re}A}\Big) \;&=:\; \delta J_0^{(2)} \;=\; \delta J_{0\,\mathrm{div}}^{(2)} + \delta J_{0\,\mathrm{fin}}^{(2)}\\
\delta J_{0\,\mathrm{div}}^{(2)} \;&=\; \frac{1}{8\pi^2}\,\xi_c\delta R(t)\Big(\frac{1}{\varepsilon}+\frac{\gamma}{2}+1+\frac{1}{2}\ln\pi\Big)\\
\delta J_{0\,\mathrm{fin}}^{(2)} \;&=\; \frac{1}{8\pi^2}\Big(-\frac{5}{4}\,\dot I(t) + I(t)\,\big(\frac{57}{24}-\frac{\nu^2}{2\sin^2\pi\nu}\big)\\
&\quad - I(t)\,\tilde m_0^2\Big(\frac{5}{2}+3\gamma-3\ln 2+\frac{3}{2}\,\psi(\tfrac{3}{2}{+}\nu)+\frac{3}{2}\,\psi(\tfrac{3}{2}{-}\nu)\Big)\\
&\quad + r(t)\,\tilde G_R(1,0) - 2I(t)\,\tilde G_R(1,2) + \dot I(t)\,\tilde G_R(2,2) - \big(\frac{\ddot I(t)}{2}+\dot I(t)\big)\,\tilde G_R(3,2)\\
&\quad - r(0)\,G_R(t;1,0) + 2I(0)\,G_R(t;1,2) - \dot I(0)\,G_R(t;2,2)\\
&\quad + \big(\frac{\ddot I(0)}{2}+\dot I(0)\big)\,G_R(t;3,2) - \int_0^t dt'\,(\dot r(t')+2r(t'))\,G_R(t{-}t';1,0)\\
&\quad + \int_0^t dt'\,\big(\frac{\dddot I(t')}{2}+3\ddot I(t')+4\dot I(t')\big)\,G_R(t{-}t';3,2)\\
&\quad + 2\sum_{n=-3}^{+1}\delta\Gamma_n\,G_R(t;0,n) - \frac{\pi^2}{2}\,\mathrm{Re}\int_0^\infty dy\,y^2 H_\nu^{(1)\,2}(y)\,H_\nu^{(2)\,2}(ya_0(t))\,\delta\Gamma^{(ii)}(ya_0(t))\Big)
\end{aligned}
\tag{62}
$$

The abbreviations $\tilde G_R(p,l) := \mathrm{Re}\,\tilde G(p,l)$ and $G_R(t;p,l) := \mathrm{Re}\,G(t;p,l)$ have been used.

Apart from the $1/\varepsilon$-pole in (62) and (63), which will be removed by renormalization, the functions $G_R(t;p,l)$ are divergent in the limit $t \to 0$. Using explicitly the asymptotic expansion (51) and the coefficients $\delta\Gamma_n$ (37) it turns out that the $t\to 0$-divergencies of the lower boundary terms in (55) cancel the $t\to 0$-divergencies from our $F_k^{(ii)}$ resp. $\delta\Gamma(k,0)$ as it has to be. It is for this reason that we need the $\delta\Gamma_n$-terms and the comparison with the adiabatic vacuum in section 3.1. In the same way the finiteness of the first and second time derivatives of $\delta J_{0\,\mathrm{fin}}^{(2)}$ needed in (24) has been checked for $t \to 0$.

$$
\begin{aligned}
\int \tilde{dk}\, \delta\Big(\frac{a^{-2}k^2}{2\,\mathrm{Re}A}\Big) \;&=:\; \delta J_2^{(2)} \;=\; \delta J^{(2)}_{2\,\mathrm{div}} + \delta J^{(2)}_{2\,\mathrm{fin}} \\
\delta J^{(2)}_{2\,\mathrm{div}} \;&=\; \frac{1}{8\pi^2}\Big(\frac{\tilde{m}_0^2}{2}\,\ddot{I}(t) + 3\tilde{m}_0^2 \dot{I}(t) - \frac{3}{2}\,(\tilde{m}_0^2{+}1)\,\xi_c \delta R(t) - \frac{5}{4}\,\xi_c \delta\dot{R}(t) - \frac{1}{4}\,\xi_c \delta\ddot{R}(t)\Big) \\
&\qquad\qquad \cdot\Big(\frac{1}{\varepsilon} + \frac{\gamma}{2} + 1 + \frac{1}{2}\ln\pi\Big) \\
\delta J^{(2)}_{2\,\mathrm{fin}} \;&=\; \frac{1}{8\pi^2}\Big(-I(t)\Big(\frac{131}{16} + \frac{49}{16}\nu^2 + \frac{\nu^4 - 97\nu^2/16}{\sin^2\pi\nu}\Big) + \dot{I}(t)\Big(\frac{5}{2} - \frac{17}{8}\nu^2 - \frac{7\,\nu^2}{8\sin^2\pi\nu}\Big) \\
&\quad - \xi_c \delta R(t)\Big(\frac{3}{48} + \frac{\nu^2}{4\sin^2\pi\nu}\Big) - \frac{5}{8}\,\xi_c \delta\dot{R}(t) \\
&\quad + \frac{15}{4}\,I(t)\,\tilde{m}_0^2\Big(\frac{3}{2} + (m^2 - \xi R_0)\Big(\frac{47}{60} + \gamma - \ln 2 + \frac{1}{2}\,\psi(\tfrac{3}{2}{+}\nu) + \frac{1}{2}\,\psi(\tfrac{3}{2}{-}\nu)\Big)\Big) \\
&\quad - r(t)\,\tilde{G}_R(1,2) + \Big(\frac{\dot{r}(t)}{2} + r(t)\Big)\,\tilde{G}_R(2,2) - \Big(\frac{\ddot{r}(t)}{4} + \frac{3}{2}\,\dot{r}(t) + 2r(t)\Big)\,\tilde{G}_R(3,2) \\
&\quad + 2I(t)\,\tilde{G}_R(1,4) - \dot{I}(t)\,\tilde{G}_R(2,4) - \Big(\frac{\dddot{I}(t)}{4} + \frac{3}{2}\,\ddot{I}(t) + 2\dot{I}(t)\Big)\,\tilde{G}_R(4,4) \\
&\quad + \Big(\frac{\ddot{I}(t)}{2} + \dot{I}(t)\Big)\,\tilde{G}_R(3,4) + \Big(\frac{I^{(\mathrm{iv})}(t)}{8} + \frac{3}{2}\,\dddot{I}(t) + \frac{11}{2}\,\ddot{I}(t) + 6\dot{I}(t)\Big)\,\tilde{G}_R(5,4) \\
&\quad + r(0)\,\tau_0^2 G_R(t;1,2) - \Big(\frac{\dot{r}(0)}{2} + r(0)\Big)\,\tau_0^2 G_R(t;2,2) \\
&\quad + \Big(\frac{\ddot{r}(0)}{4} + \frac{3}{2}\,\dot{r}(0) + 2r(0)\Big)\tau_0^2 G_R(t;3,2) - 2I(0)\,\tau_0^2 G_R(t;1,4) + \dot{I}(0)\,\tau_0^2 G_R(t;2,4) \\
&\quad - \Big(\frac{\ddot{I}(0)}{2} + \dot{I}(0)\Big)\,\tau_0^2 G_R(t;3,4) + \Big(\frac{\dddot{I}(0)}{4} + \frac{3}{2}\,\ddot{I}(0) + 2\dot{I}(0)\Big)\,\tau_0^2 G_R(t;4,4) \\
&\quad - \Big(\frac{I^{(\mathrm{iv})}(0)}{8} + \frac{3}{2}\,\dddot{I}(0) + \frac{11}{2}\,\ddot{I}(0) + 6\dot{I}(0)\Big)\,\tau_0^2 G_R(t;5,4) \\
&\quad + \int_0^t dt'\,\Big(\frac{\dddot{r}(t')}{4} + 3\,\ddot{r}(t') + 11\,\dot{r}(t') + 12\,r(t')\Big)\,\tau_0^2 G_R(t{-}t';3,2) \\
&\quad - \int_0^t dt'\,\Big(\frac{I^{(\mathrm{v})}(t')}{8} + \frac{5}{2}\,I^{(\mathrm{iv})}(t') + \frac{35}{2}\,\dddot{I}(t') + 50\,\ddot{I}(t') + 48\,\dot{I}(t')\Big)\,\tau_0^2 G_R(t{-}t';5,4) \\
&\quad - 2\sum_{n=-3}^{+1} \delta\Gamma_n \tau_0^2 G_R(t;0,n{+}2) - \frac{\pi^2}{2}\,\mathrm{Re}\int_0^\infty dy\, y^4 H_\nu^{(1)^2}(y)\, H_\nu^{(2)^2}(ya_0(t))\,\delta\Gamma^{(ii)}(ya_0(t))\Big)
\end{aligned}
\qquad (63)
$$

3.3 Renormalization

Our renormalization scheme consists in the subtraction of De Witt-Schwinger terms as was explained in section 2. Again we linearize with respect to the deviation from the de Sitter – Bunch-Davies system. With the aid of appendix B equation (16) leads to:

$$
\begin{aligned}
\delta\rho_{\mathrm{DS\,div}} &= \frac{1}{16\pi^2}\Big(\frac{1}{\varepsilon}-\frac{1}{2}\Big(\gamma+\ln\frac{m^2}{4\pi}\Big)\Big) \\
&\quad\cdot\Big(\frac{-4\,m^2}{d-1}\,(\xi-\tfrac{1}{6})\,\delta G_{00}+\frac{1}{90}\Big(\delta H_{00}-\delta\,^{(2)}H_{00}\Big)+(\xi-\tfrac{1}{6})^2\delta\,^{(1)}H_{00}\Big) \\
&= \frac{1}{16\pi^2}\Big(\frac{1}{\varepsilon}-\frac{1}{2}\Big(\gamma+\ln\frac{m^2}{4\pi}\Big)\Big)\Big(\frac{-4\,m^2}{d-1}\,(\xi-\tfrac{1}{6})\,d(d-1)\,\dot I \\
&\qquad+\frac{1}{90}\,d(d-3)\Big(\dddot I+d\,\ddot I+2(d-2)\,\dot I\Big) \\
&\qquad-(\xi-\tfrac{1}{6})^2 d^2\Big(4\,\dddot I+4d\,\ddot I+2(d+1)(d-3)\,\dot I\Big)\Big) \qquad (64)\\
\delta p_{\mathrm{DS\,div}} &= \frac{1}{16\pi^2}\Big(\frac{1}{\varepsilon}-\frac{1}{2}\Big(\gamma+\ln\frac{m^2}{4\pi}\Big)\Big)\Big(\frac{-4\,m^2}{d-1}\,(\xi-\tfrac{1}{6})\,(1-d)\Big(\ddot I+d\,\dot I\Big) \\
&\qquad+\frac{1}{90}\,(3-d)\Big(I^{(\mathrm{iv})}+2d\,\dddot I+(d^2+2d-4)\,\ddot I+2d(d-2)\,\dot I\Big) \\
&\qquad+(\xi-\tfrac{1}{6})^2 d\Big(4\,I^{(\mathrm{iv})}+8d\,\dddot I+(6d^2-4d-6)\,\ddot I+2d(d-3)(d+1)\,\dot I\Big)\Big)
\end{aligned}
$$

Together with (24), (62) and (63) we finally obtain the components of the renormalized energy momentum tensor expectation value:

$$
\begin{aligned}
\delta\rho_{\rm ren} &= \delta\rho - \delta\rho_{\rm DS\,div} \\
&= \xi\,(\delta R_{00} - \delta R)\,J_0^{(2)} + (m^2 + \xi\,d^2 + d\,(\xi + \tfrac14)\,\partial_t + \tfrac14\,\partial_t^2)\,(\delta J^{(2)}_{0\,\rm div} + \delta J^{(2)}_{0\,\rm fin}) \\
&\quad + \delta J^{(2)}_{2\,\rm div} + \delta J^{(2)}_{2\,\rm fin} - \delta\rho_{\rm DS\,div} \\
&= \frac{1}{8\pi^2}\Big(\frac{m^2}{4}\,\ddot I + m^2(3\,\xi_c + \tfrac73)\,\dot I + 2\xi_c\Big(\dddot I + 3\ddot I\Big) + \frac{27}{2}\,\xi_c\ddot I + 12\,\xi_c(3\xi_c + 5)\,\dot I \\
&\quad + \frac{1}{60}\Big(\dddot I + 3\ddot I + 2\dot I\Big) - 36\,\xi_c^2\dot I \\
&\quad + \frac{\tilde m_0^2}{2}\,\xi\,(\delta R_{00} - \delta R)\Big(\psi(\tfrac32 + \nu) + \psi(\tfrac32 - \nu) - 1 - \ln m^2\Big) \\
&\quad + (m^2 + 9\,\xi + 3\,(\xi + \tfrac14)\,\partial_t + \tfrac14\,\partial_t^2)\Big(\xi_c\delta R\Big(\gamma + 1 + \frac12\ln\frac{m^2}{4}\Big) + 8\pi^2\delta J^{(2)}_{0\,\rm fin}\Big) \\
&\quad + \Big(\frac{\tilde m_0^2}{2}\ddot I + 3\tilde m_0^2\dot I - \frac32\,(\tilde m_0^2 + 1)\,\xi_c\delta R - \frac54\,\xi_c\delta\dot R - \frac14\,\xi_c\delta\ddot R\Big)\Big(\gamma + 1 + \frac12\ln\frac{m^2}{4}\Big) \\
&\quad + 8\pi^2\delta J^{(2)}_{2\,\rm fin}\Big)
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\delta p_{\rm ren} &= \delta p - \delta p_{\rm DS\,div} \\
&= \frac{1}{8\pi^2}\Big(m^2\Big(\frac{5}{36}\,\ddot I + \dot I\Big) + m^2\xi_c\dot I + 12\,\xi_c^2\Big(\ddot I + 4\dot I\Big) + \xi_c\Big(\frac{I^{(\rm iv)}}{3} + 3\,\dddot I + \frac{21}{2}\,\ddot I + 20\,\dot I\Big) \\
&\quad - \frac{1}{180}\Big(I^{(\rm iv)} + 6\,\dddot I + 11\,\ddot I + 6\,\dot I\Big) \\
&\quad + \frac{\tilde m_0^2}{2}\,\xi\,\delta(a^{-2}R_{11})\Big(\psi(\tfrac32 + \nu) + \psi(\tfrac32 - \nu) - 1 - \ln m^2\Big) \\
&\quad + (\xi\,a^{-2}R_{11} + (\tfrac34 - 2\xi)\,\partial_t + (\tfrac14 - \xi)\,\partial_t^2)\Big(\xi_c\delta R\Big(\gamma + 1 + \frac12\ln\frac{m^2}{4}\Big) + 8\pi^2\delta J^{(2)}_{0\,\rm fin}\Big) \\
&\quad + \frac13\Big(\frac{\tilde m_0^2}{2}\ddot I + 3\tilde m_0^2\dot I - \frac32\,(\tilde m_0^2 + 1)\,\xi_c\delta R - \frac54\,\xi_c\delta\dot R - \frac14\,\xi_c\delta\ddot R\Big)\Big(\gamma + 1 + \frac12\ln\frac{m^2}{4}\Big) \\
&\quad + \frac{8\pi^2}{3}\,\delta J^{(2)}_{2\,\rm fin}\Big)
\end{aligned}
\tag{66}
$$

According to $d=3$, $\xi_c = \xi - \frac16$.

All divergencies have cancelled and the final results (65) and (66) are finite. This fact can be regarded a non-trivial check on the calculation.

Contrary to the other terms the $\delta J^{(2)}_{l\,\rm fin}$'s are non-local functionals of $r(t)$ and $I(t)$, because they contain convolution integrals as well as the initial data at $t = 0$.

The G-functions appearing in the $\delta J^{(2)}_{l\,\rm fin}$'s in (62) and (63) are defined as convergent series and are therefore well suited for a numerical computation.

4 The linearized Einstein equations

Restricting on FRW spacetimes the semiclassical Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R + \Lambda\, g_{\mu\nu} \;=\; 8\pi G_N \, \langle T_{\mu\nu} \rangle_{\mathrm{ren}} \tag{67}$$

are containing two independent components. They read (remember $H(t) := \dot{a}/a = H_0 + \dot{I}(t)$)

$$3\, H^2 + \Lambda \;=\; 8\pi G_N \, \rho \tag{68}$$

$$2\, \dot{H} + 3\, H^2 + \Lambda \;=\; -8\pi G_N \, p\, . \tag{69}$$

The index "ren" at ρ and p will be suppressed and the Hubble constant H_0 will be explicitly written out in this section.

If the sources ρ and p are specified in advance equation (68) (the 00-component of (67)) is no dynamical equation of motion but a constraint on the initial data. It plays the role of the Poisson equation in electrodynamics. Here instead the sources are themselves functionals of the metric and are reacting on its changes. Therefore equation (68) is a dynamical equation in the present case.

Neither the dimensional regularization nor our renormalization scheme are spoiling the covariant energy momentum conservation:

$$D_\mu \langle T^{\mu 0} \rangle_{\mathrm{ren}} \;=\; \dot{\rho} + 3H(\rho + p) \;=\; 0 \tag{70}$$

On that account the equations (68) and (69) are not independent: Every solution of (68) is also a solution to (69). Therefore, only (68) is considered in the following. In its linearized form it reads:

$$6\, \frac{\dot{I}(t)}{H_0} \;=\; \frac{8\pi G_N}{H_0^2}\, \delta\rho[I;t] \tag{71}$$

Due to the convolutions contained in $\delta\rho$ this is a linear integro-differential equation and may be conveniently solved by Laplace transformation.

4.1 Laplace transformation

The Laplace transform of a function $f(t)$ will be denoted by $\mathcal{L}[f;s]$ or $\widehat{f}(s)$:

$$\mathcal{L}[f;s] \ := \ \widehat{f}(s) \ := \ \int_0^\infty dt\, f(t)\, e^{-st} \tag{72}$$

Utilizing

$$\widehat{\dot{f}}(s) \ = \ s\widehat{f}(s) - f(0)\,, \qquad \widehat{f^{(n)}}(s) \ = \ s^n\widehat{f}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)\,,$$

$$\mathcal{L}\Big[\int_0^t dt'\, f(t')\, g(t-t')\,;s\Big] \ = \ \widehat{f}(s)\cdot\widehat{g}(s) \tag{73}$$

one is able to compute the Laplace transform $\widehat{\delta\rho}(s)$ of $\delta\rho$ (65). For the $\delta J^{(2)}_{\rm fin}$'s we need the transforms $\widehat{G_R}(s;p,l)$ and $\widehat{\tau_0^2 G_R}(s;p,l)$ of the functions $G_R(t;p,l)$ and $\tau_0^2(t)G_R(t;p,l)$, which were defined in (43) in terms of three generalized hypergeometric series ${}_4\bar{F}_3(\ldots;\ldots;\tau_0^2(t))$. Their Laplace transforms are again hypergeometric series:

$$\begin{aligned}
\mathcal{L}\Big[\tau_0^{2\alpha}\,{}_4\bar{F}_3(\ldots;\ldots;\tau_0^2(t))\,;s\Big] \ &= \ \mathcal{L}\Big[e^{-2\alpha H_0 t}\sum_{n=0}^{\infty} Q_n(\ldots;\ldots)\, e^{-2H_0 tn}\,;s\Big] \\
&= \ \sum_{n=0}^{\infty} \frac{Q_n(\ldots;\ldots)}{s+2H_0 n+2\alpha H_0} \\
&= \ \frac{1}{2H_0}\,{}_5\bar{F}_4\Big(\frac{s}{2H_0}+\alpha,\ldots;\frac{s}{2H_0}+\alpha+1,\ldots;1\Big)
\end{aligned} \tag{74}$$

The particular functions ${}_4\bar{F}_3(\ldots;\ldots;\tau_0^2(t))$ in the $\delta J^{(2)}_{\rm fin}$'s (62) and (63) are singular for $t\to 0$, so that the corresponding series (74) do not converge. Therefore all ${}_4\bar{F}_3$-series in $\delta J^{(2)}_{0\,\rm fin}$ respectively $\delta J^{(2)}_{2\,\rm fin}$ have to be added term by term before summing up the series. The complete $\delta J^{(2)}_{\rm fin}$'s are well behaved for $t\to 0$ (see the comments after (62)), and the term by term addition of the series (74) will be convergent.
The series $\widehat{G_R}(s;1,0)$, $\widehat{G_R}(s;3,2)$ and $\widehat{G_R}(s;5,4)$ appearing in the convolutions are convergent by themselves. For large s they are of the order of $s^{-1}(1+\ln s)$.
The Laplace transform $\widehat{\delta\rho}(s)$ may be cast into the form

$$\widehat{\delta\rho}(s) \ =: \ \frac{H_0^4}{8\pi^2}\Big(\delta\rho_{\widehat{I}}\Big(\frac{s}{H_0}\Big)\,\widehat{I}(s) + \sum_{n=0}^{2} f_n\Big(\frac{s}{H_0}\Big)\,\frac{I^{(n)}(0)}{H_0^{n+1}} + \frac{1}{H_0}\,g\Big(\frac{s}{H_0}\Big)\Big) \tag{75}$$

with certain functions $\delta\rho_{\widehat{I}}(s/H_0)$, $f_n(s/H_0)$ and $g(s/H_0)$. The function g contains the $\delta\Gamma(k,0)$ - resp. $\delta\Gamma_n$ - and $\delta\Gamma^{(ii)}(k)$ -contributions.

The explicit calculation shows[2] that the $\dddot{I}(0)$ - and the $I^{(\mathrm{iv})}(0)$ -terms drop out in (75). Only up to second derivatives of $I(t)$ at time $t=0$ are appearing in $\widehat{\delta\rho}(s)$ and thereby in (77) (apart from the $\delta\Gamma_n$, see below). This means that we have to specify the initial data $\delta\Gamma(k,0)$, $I(0)$, $\dot{I}(0)$ and $\ddot{I}(0)$ in order to fix the solution of (71). Regarding $\delta\Gamma(k,0)$ we only have $\delta\Gamma^{(ii)}(k)$ at our disposal (see (30) and afterwards), whereas (37) has to be used for the $\delta\Gamma_n$. The values of $\dddot{I}(0)$ and $I^{(\mathrm{iv})}(0)$ required in (37) may be obtained from equation (71) and its derivative at time $t=0$. However, due to the smallness of Newton's constant it is more advantageous to specify $\dddot{I}(0)$ instead of $\dot{I}(0)$ and to determine $\dot{I}(0)$ from (71) at $t=0$. Then it can be easily granted that all $I^{(n)}$ are small at $t=0$ and that the linearization is justified.

We want to save the explicit statement of the lengthy and in the following unessential functions $f_n(s)$ and $g(s)$, but at least $\delta\rho_{\widehat{I}}(s)$ should be given. The expression has been left in a somewhat uncompactified shape, because there is presumably not much shortening to gain:

[2]This has to prove true since (65) results from (24) and (64).

$$
\begin{aligned}
\delta\rho_{\widehat{I}}(s) \;=\;& \frac{m^2}{4H_0^2}\, s^2 + \frac{m^2}{H_0^2}\,(3\xi_c{+}\tfrac{7}{3})\, s + 2\xi_c\Big(s^3+3s^2\Big) + \frac{27}{2}\,\xi_c s^2 + 12\,\xi_c(3\xi_c{+}5)\, s \\
&+ \frac{1}{60}\Big(s^3+3s^2+2s\Big) - 36\,\xi_c^2 s \\
&+ \frac{3\tilde m_0^2}{2H_0^2}\,\xi\,\Big(s^2+6s\Big)\Big(\psi(\tfrac{3}{2}{+}\nu)+\psi(\tfrac{3}{2}{-}\nu)-1-\ln\frac{m^2}{H_0^2}\Big) \\
&+ \Big(\frac{m^2}{H_0^2}+9\,\xi+3\,(\xi{+}\tfrac{1}{4})\, s+\tfrac{1}{4}\, s^2\Big)\Big(-6\xi_c(s^2+4s)\Big(\gamma+1+\frac{1}{2}\ln\frac{m^2}{4H_0^2}\Big) \\
&\quad - \frac{5}{4}\, s + \frac{57}{24} - \frac{\nu^2}{2\sin^2\pi\nu} - \frac{\tilde m_0^2}{H_0^2}\Big(\frac{5}{2}+3\gamma-3\ln 2+\frac{3}{2}\,\psi(\tfrac{3}{2}{+}\nu)+\frac{3}{2}\,\psi(\tfrac{3}{2}{-}\nu)\Big) \\
&\quad + \Big(\frac{s^2}{2}+\frac{s}{2}-6\xi_c(s^2+4s)\Big)\,\tilde G_R(1,0) \\
&\quad - 2\,\tilde G_R(1,2) + s\,\tilde G_R(2,2) - \Big(\frac{s^2}{2}+s\Big)\,\tilde G_R(3,2) \\
&\quad - (s+2)\Big(\frac{s^2}{2}+\frac{s}{2}-6\xi_c(s^2+4s)\Big)\,\widehat{G_R}(s;1,0) + \Big(\frac{s^3}{2}+3s^2+4s\Big)\,\widehat{G_R}(s;3,2)\Big) \\
&+ \Big(\frac{\tilde m_0^2}{2H_0^2}\, s^2 + 3\,\frac{\tilde m_0^2}{H_0^2}\, s + 6\xi_c(s^2+4s)\Big(\frac{3}{2}\,\Big(\frac{\tilde m_0^2}{H_0^2}+1\Big)+\frac{5}{4}\, s+\frac{1}{4}\, s^2\Big)\Big) \\
&\quad \cdot \Big(\gamma+1+\frac{1}{2}\ln\frac{m^2}{4H_0^2}\Big) - \frac{131}{16} - \frac{49}{16}\nu^2 - \frac{\nu^4-97\nu^2/16}{\sin^2\pi\nu} \\
&+ \Big(\frac{5}{2}-\frac{17}{8}\nu^2-\frac{7\,\nu^2}{8\sin^2\pi\nu}\Big)\, s + \Big(\frac{3}{8}+\frac{3\,\nu^2}{2\sin^2\pi\nu}+\frac{15}{4}\, s\Big)\,\xi_c(s^2+4s) \\
&+ \frac{15\tilde m_0^2}{4H_0^2}\Big(\frac{3}{2}+\Big(\frac{\tilde m_0^2}{H_0^2}+2\Big)\Big(\frac{47}{60}+\gamma-\ln 2+\frac{1}{2}\,\psi(\tfrac{3}{2}{+}\nu)+\frac{1}{2}\,\psi(\tfrac{3}{2}{-}\nu)\Big)\Big) \\
&+ \Big(\frac{s^2}{2}+\frac{s}{2}-6\xi_c(s^2+4s)\Big)\Big(-\tilde G_R(1,2)+\Big(\frac{s}{2}+1\Big)\,\tilde G_R(2,2) \\
&\quad - \Big(\frac{s^2}{4}+\frac{3}{2}\, s+2\Big)\,\tilde G_R(3,2)\Big) + 2\,\tilde G_R(1,4) - s\,\tilde G_R(2,4) + \Big(\frac{s^2}{2}+s\Big)\,\tilde G_R(3,4) \\
&- \Big(\frac{s^3}{4}+\frac{3}{2}\, s^2+2s\Big)\,\tilde G_R(4,4) + \Big(\frac{s^4}{8}+\frac{3}{2}\, s^3+\frac{11}{2}\, s^2+6s\Big)\,\tilde G_R(5,4) \\
&+ \Big(\frac{s^3}{4}+3\, s^2+11\, s+12\Big)\Big(\frac{s^2}{2}+\frac{s}{2}-6\xi_c(s^2+4s)\Big)\,\widehat{\tau_0^2 G_R}(s;3,2) \\
&- \Big(\frac{s^5}{8}+\frac{5}{2}\, s^4+\frac{35}{2}\, s^3+50\, s^2+48\, s\Big)\,\widehat{\tau_0^2 G_R}(s;5,4)
\end{aligned} \tag{76}
$$

For a numerical computation of $\delta\rho_{\widehat{I}}(s)$ we need the functions $\tilde G_R(p,l)$ ((43) and (59)) and $\widehat{G_R}(s;p,l)$ (74). According to the recurrence properties of the Q_n (48) they can be easily approximated by a direct numerical summation of their cor-

responding series up to the N-th term (provided that they are not alternating). Since the terms of the series behave like n^{-2} for large n, one encounters an error of the order of N^{-1} due to the truncation (in case of $\widehat{G_R}(s;p,l)$, $N > s$ should apply). In order that this really comes true, one has to work with a great numerical precision. For an example consider the most difficult case $\tilde{G}_R(1,4)$:

$$\tilde{G}_R(1,4) \;\sim\; \sum_n \Big(\mathcal{O}(n^3) - \mathcal{O}(n^3)\Big) \;\stackrel{!}{\sim}\; \sum_n \mathcal{O}(n^{-2})$$

Two terms of the order of n^3 calculated independently have to cancel with an accuracy of n^{-2} in the large-n terms of (59). Hence a relative accuracy of N^{-5} is required for the calculation of these terms, if the summation of the series up to $n = N$ is supposed to be sensible. Choosing $N = 10000$, a relative precision of 10^{-20} is required!

For that reason we used REAL*16 variables (29 significant digits). The precision approximation for the gamma-function of Lanczos [17] has been extended to a relative accuracy $< 10^{-25}$ in the whole complex plane and was used for calculating the Q_0.

4.2 Stability

The Laplace transform of equation (71) reads ($M_{Pl} = G_N^{-1/2}$)

$$6\,\frac{s\widehat{I}(s) - I(0)}{H_0} \;=\; \frac{H_0^2}{\pi M_{Pl}^2}\Big(\delta\rho_{\widehat{I}}\Big(\frac{s}{H_0}\Big)\,\widehat{I}(s) + \sum_{n=0}^{2} f_n\Big(\frac{s}{H_0}\Big)\,\frac{I^{(n)}(0)}{H_0^{n+1}} + \frac{1}{H_0}\,g\Big(\frac{s}{H_0}\Big)\Big) \quad (77)$$

and is solved by

$$\widehat{I}(s) \;=\; \frac{6\,\frac{I(0)}{H_0} + \frac{H_0^2}{\pi M_{Pl}^2}\Big(\sum_{n=0}^2 f_n\big(\frac{s}{H_0}\big)\,\frac{I^{(n)}(0)}{H_0^{n+1}} + \frac{1}{H_0}\,g\big(\frac{s}{H_0}\big)\Big)}{6\,\frac{s}{H_0} - \frac{H_0^2}{\pi M_{Pl}^2}\,\delta\rho_{\widehat{I}}\big(\frac{s}{H_0}\big)}\,. \quad (78)$$

Inverting the Laplace transform we get the exact general solution $I(t)$ of the linearized "backreaction problem":

$$I(t) \;=\; \int_{-i\infty+\alpha}^{+i\infty+\alpha} \frac{ds}{2\pi i}\,\widehat{I}(s)\,e^{st} \quad (79)$$

The contour of integration runs on the right of all poles of the integrand ($\alpha > \ldots$). The numerator of $\widehat{I}$ is of the order of $s^2(1 + \ln s)$ for large $|s|$ (on the negative real axis between its poles) and the denominator is of the order of $s^3(1+\ln s)$ (see below). Therefore one can close the contour of integration for $t > 0$ by a sequence of semicircles in the left half complex plane extended by α to the right with their centres being located at the origin (compare with the reflected image of figure A.1). Their radii have to be choosen in such a way that they do not come too close to the poles of the numerator of $\widehat{I}$ on the negative real axis. Then the contributions of these semicircles to the integral vanish in the limit of infinite radii. Hence the integral (79) is equal to the sum over the residues of all poles of its integrand.

We are not going to perform the back-transformation explicitly, but investigate the question of the existence of any instabilities. Due to the factor e^{st} growing terms in $I(t)$ for large t are only possible through poles at $s = s_0$ with $\mathrm{Re}\, s_0 \geq 0$. By the functions $\widehat{G_R}(s; p, l)$ the numerator of $\widehat{I}$ has infinitely many poles (see (74)), but they are all located in the left half complex plane. Thus only the zeros of the denominator of $\widehat{I}$ are of interest to us.

In the following we are seeking solutions of the equation

$$6\,\frac{s}{H_0} \;=\; \frac{H_0^2}{\pi M_{Pl}^2}\,\delta\rho_{\widehat{I}}\Big(\frac{s}{H_0}\Big) \tag{80}$$

with $\mathrm{Re}\, s \geq 0$. Note that the factor H_0^2/M_{Pl}^2 has to be small compared to 1. This is a necessary condition for the applicability of the semiclassical theory. In the new inflationary universe scenario for example we have $H_0 \sim 10^{11}\,\mathrm{GeV}$ ($M_{Pl} = 10^{19}\,\mathrm{GeV}$, $H(\mathrm{today}) = 10^{-42}\,\mathrm{GeV}$).

First we will consider two hypothetical situations where instabilities could occur:

<u>(i) $\delta\rho_{\widehat{I}}(s) \overset{s\to 0}{\longrightarrow} \beta > 0$</u> In this case a solution of (80) would be

$$\frac{s}{H_0} \;\simeq\; \frac{\beta}{6\pi}\,\frac{H_0^2}{M_{Pl}^2}$$

corresponding to an instability on a large time scale s^{-1}. However the numerical calculations show that β is always of the order of N^{-1}, which is the error due to

the truncation of the hypergeometric series at the N-th term. This observation leads to the conjecture $\beta = 0$, which can be proven by a simple consideration: The constant terms $\sim s^0$ in $\delta\rho_{\widehat{I}}(s)$ are coming from contributions to $\delta\rho$ proportional to $I(t)$ which are present even for a constant $I(t)$. A constant I in the scale factor (23) of the metric (2) can be removed by the coordinate transform $t' = t$, $\vec{x}' = (1+I)^{1/2}\vec{x}$, for which the 00-component of the energy momentum tensor (21) behaves like a scalar and remains unchanged! Hence $s = 0$ must be a solution of (80), but it is a pure gauge mode (coordinate transform).

With $\delta\rho_{\widehat{I}}(0) = 0$ we have indirectly proven the following two, on this level remarkable identities (using (76)):

$$2\,\tilde{G}_R(1,2) \;=\; \frac{57}{24} - \frac{\nu^2}{2\sin^2\pi\nu} - \frac{\tilde{m}_0^2}{H_0^2}\Big(\frac{5}{2} + 3\gamma - 3\ln 2 + \frac{3}{2}\,\psi(\tfrac{3}{2}+\nu) + \frac{3}{2}\,\psi(\tfrac{3}{2}-\nu)\Big)$$

$$\begin{aligned} 2\,\tilde{G}_R(1,4) \;=\; & \frac{131}{16} + \frac{49}{16}\nu^2 + \frac{\nu^4 - 97\nu^2/16}{\sin^2\pi\nu} \\ & - \frac{15\tilde{m}_0^2}{4H_0^2}\Big(\frac{3}{2} + \Big(\frac{\tilde{m}_0^2}{H_0^2} + 2\Big)\Big(\frac{47}{60} + \gamma - \ln 2 + \frac{1}{2}\,\psi(\tfrac{3}{2}+\nu) + \frac{1}{2}\,\psi(\tfrac{3}{2}-\nu)\Big)\Big) \end{aligned}$$

It is also possible to prove these identities on the level of equation (29), but that will be skipped here.

The numerical result $\beta = 0$ can be considered a non-trivial check for the numerical as well as analytical calculation.

<u>(ii) $\delta\rho_{\widehat{I}}(s) \overset{s\gg 1}{\simeq} \beta s^4$</u> This behaviour would lead to the solution

$$s \;\simeq\; \Big(\frac{6\pi}{\beta}\Big)^{1/3}\Big(\frac{H_0}{M_{Pl}}\Big)^{1/3} M_{Pl}$$

and thereby to an instability on a short time scale s^{-1} (compared with H_0^{-1}) that still lies in the semiclassical region. However, again the numerical investigation first showed that the s^4-terms in $\delta\rho_{\widehat{I}}$ cancel each other:

$$\begin{aligned} &\tfrac{1}{8}\Big((1-12\xi_c)(\tilde{G}_R(1,0) - s\widehat{G_R}(s;1,0)) - (\tilde{G}_R(3,2) - s\widehat{G_R}(s;3,2)) \\ &\quad -(1-12\xi_c)(\tilde{G}_R(3,2) - s\widehat{\tau_0^2 G_R}(s;3,2)) + \tilde{G}_R(5,4) - s\widehat{\tau_0^2 G_R}(s;5,4)\Big) \\ &\sim \mathcal{O}\Big(s^{-1}(1+\ln s)\Big) \end{aligned}$$

The reason is that fourth derivatives $I^{(\mathrm{iv})}(t)$ do emerge only in the convolution integrals (40) from the singular behaviour of the kernels $J_l^{(4)}$. Using (53) one sees that in fact in (24) the corresponding terms from $\frac{1}{4}\partial_t^2 \int \tilde{\tilde{dk}}\, \delta(2\,\mathrm{Re}A)^{-1}$ and from $\int \tilde{\tilde{dk}}\, \delta(a^{-2}k^2/2\,\mathrm{Re}A)$ cancel.

More easily it follows from the energy momentum conservation $\dot{\delta\rho}+3H_0(\delta\rho+\delta p) = 0$, that $\delta\rho$ has to contain one time derivative less than δp ($\dot\rho_0 = 0 = \rho_0 + p_0$, see (21)).

Summarizing we did not find an instability but again a non-trivial check on the numerical as well as analytical calculation.

Now it's time to stop making hypothesises and to look at the real behaviour of

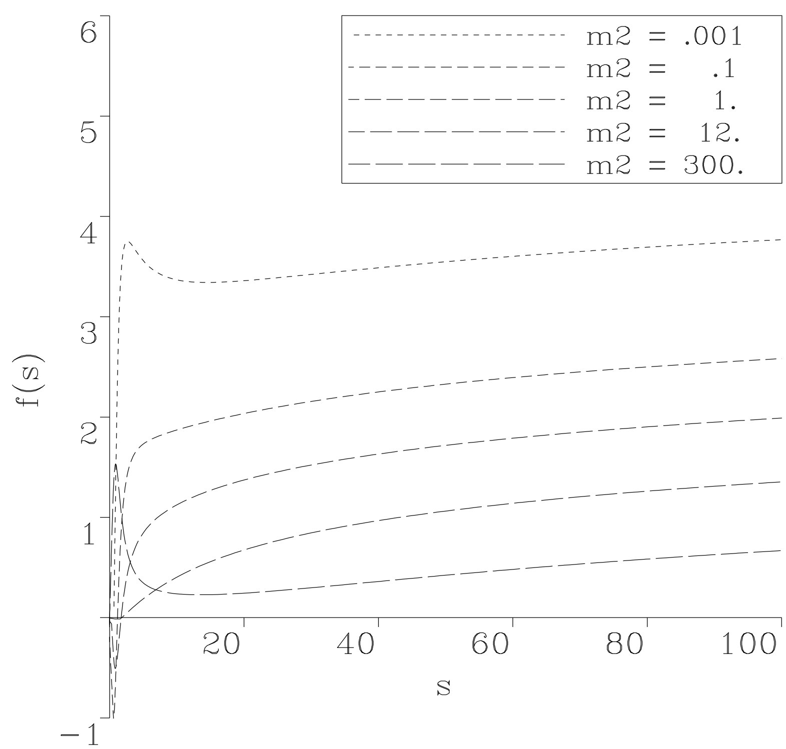

Figure 1: The function $f(s) := \delta\rho_{\hat{I}}(s)/(s^3+1)$ for $\xi = 0$ and certain values of $m2 := m^2/H_0^2$. For $m2 = 300$ a peak appears on the left which is due to numerical inaccuracies. Contrary to the rest of the curve it depends strongly on the numerical precision N^{-1}.

$\delta\rho_{\hat{I}}(s)$. The function $\delta\rho_{\hat{I}}(s)/(s^3+1)$ has been plotted in the figures 1 and 2 for two values of ξ and different values of m^2 for real positive s. The curves are growing only logarithmically for large s.

It has been verified and can also be seen in (76) that the slope of $\delta\rho_{\hat{I}}$ near $s=0$ does not reach an excessive large numerical value. Since H_0^2/M_{Pl}^2 is small, a large s only can therefore solve equation (80). According to our considerations so far and to the numerical investigations $\delta\rho_{\hat{I}}(s)$ behaves like $\alpha s^3(\ln s+\beta)$ for large s, so that (80) has the following solution:

$$s \;=\; \Big(\frac{6\pi}{\alpha'}\Big)^{1/2} M_{Pl} \tag{81}$$

α' depends on α and β without receiving overwhelming large numerical values. Hence this solution would lead to an instability on the Planck time scale, which is

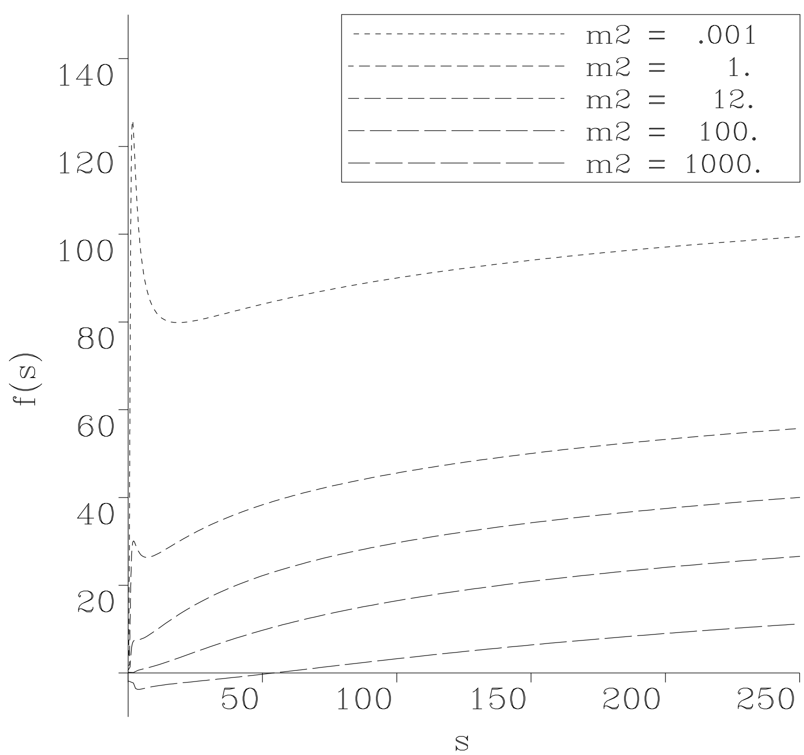

Figure 2: The function $f(s) := \delta\rho_{\hat{I}}(s)/(s^3+1)$ for $\xi=1$ and certain values of $m2 := m^2/H_0^2$

apart from the region of validity of the semiclassical theory. Within our semiclassical treatment we are not able to conclude for an instability on the Planck scale. The investigation of this region remains a subject for a future quantum theory of gravity.

For a constant ratio m^2/H_0^2 the solution (81) does not depend on H_0 (apart from logarithmic terms), so that it would be present even for very small H_0. Merely the existence of our present universe seems therefore to exclude an instability of this kind within a complete, applying theory. Probably it is just an artefact of the semiclassical treatment, as was argued for example in reference [7].

For the practical use of our general semiclassical solution it is always possible to avoid this Planck mode by requiring the numerator of (78) to have a zero at the same value (81), too. This condition is one constraint in the space of initial data which were discussed in section 4.1. The general solution then consists of an infinite series of exponentially damped modes due to the poles from (78) with $\mathrm{Re}\, s < 0$ and a constant mode from the $s = 0$ pole. The last one is the only one to survive for late times, but it corresponds just to a spatial rescaling of the underlying de Sitter spacetime with no influence on physical observables (like its curvature for example).

5 Summary and conclusions

We have linearized the semiclassical system of Schrödinger equation and Einstein equations (1) around the de Sitter – Bunch-Davies solution in order to investigate the stability of this solution against small fluctuations of the quantum state and small, spatially homogeneous and isotropic perturbations of the k=0 FRW de Sitter metric. This linearization in the sense of a stability analysis is the only approximation appearing in the present work.

The condition of finite energy density for the initial quantum state has been discussed and the Schrödinger equation was completely solved. The expectation value

of the energy momentum tensor has been calculated as a functional of the metric perturbation. The momentum integrations have been carried out analytically. A procedure for the isolation of divergencies was developed. These were removed by a renormalization through the subtraction of De Witt-Schwinger terms.

The linearized semiclassical Einstein equations became a linear integro-differential equation for the metric perturbation. After finding out the initial data which can be specified the general solution was obtained in terms of its Laplace transform. This Laplace transform has been analyzed analytically as well as numerically using the necessary high numerical precision. The general solution contains only two potential instabilities: a constant mode which corresponds just to a constant spatial rescaling of the underlying de Sitter spacetime i.e. a pure gauge, and an instability on the Planck time scale which is outside of the scope of a semiclassical theory.

Thus we have shown that de Sitter spacetime and Bunch-Davies vacuum are stable within our semiclassical theory!

The same result was obtained in reference [7] for the special case of minimal coupling $\xi = 0$. Since there the momentum integrals are not evaluated explicitly, complicated estimates were necessary in order to achieve this and there is no numerical analysis.

Our conclusion above is in contradiction with some claims existing in the literature. In particular in ref. [4] an instability on the Hubble time scale H_0^{-1} was found. They use indeed a different coordinate system (k=+1 FRW), but the coordinate lines $t =$ const., on which the spatially homogeneous initial data of the perturbation are specified, tend to coincide within the k=0 and k=+1 FRW parametrizations for late times $t \to \infty$. Therefore the answer to the stability issue should be the same. Some criticism on appendix A of ref. [4] is already contained in ref. [7]. Furthermore the diverging perturbation $\sigma(\eta)$ of the conformal factor found in ref. [4, section 5] is very near to a pure gauge mode. It can be transformed into our $I(t)$ remaining small for all times and contains no physical divergence.

Finally a few possibilities for a continuation of this work should be mentioned:

Fermi fields could be included, but presumably they wouldn't alter the stability argumentation. The investigation of the minimally coupled massless case would be interesting, because the renormalized two point function in de Sitter spacetime (see refs. [12, 18, 19]) as well as our result (76) are containing a logarithmic infrared divergence (the linear infrared divergencies of the ψ- and G-functions cancel). At last one could try to access the region of quantum gravity via the Wheeler-De Witt equation.

I would like to thank W. Buchmüller for many helpful discussions in the course of this work and D. Litim for many valuable comments on the manuscript.

A Integrals of products of Hankel functions

The definite integral of a product of two Hankel functions can be evaluated using the Weber-Schafheitlin integral [20]. One obtains:

$$\int_0^\infty dx\, x^\lambda H_\nu^{(1)}(x)\, H_\nu^{(2)}(x) \;=\; \pi^{-\frac{5}{2}}\, \Gamma\Big(-\frac{\lambda}{2}\Big)\, \Gamma\Big(\frac{1+\lambda}{2}\Big) \cos\nu\pi\, \Gamma\Big(\frac{1+\lambda}{2}+\nu\Big)\, \Gamma\Big(\frac{1+\lambda}{2}-\nu\Big) \tag{A.1}$$

$$\int_0^\infty dx\, x^\lambda H_\nu^{(1)^2}(x) \;=\; -i\,\pi^{-\frac{3}{2}}\, e^{-i\pi(\nu-\frac{\lambda}{2})}\, \frac{\Gamma(\frac{1+\lambda}{2})}{\Gamma(1+\frac{\lambda}{2})}\, \Gamma\Big(\frac{1+\lambda}{2}+\nu\Big)\, \Gamma\Big(\frac{1+\lambda}{2}-\nu\Big) \tag{A.2}$$

$$\int_0^\infty dx\, x^\lambda H_\nu^{(2)^2}(x) \;=\; i\,\pi^{-\frac{3}{2}}\, e^{i\pi(\nu-\frac{\lambda}{2})}\, \frac{\Gamma(\frac{1+\lambda}{2})}{\Gamma(1+\frac{\lambda}{2})}\, \Gamma\Big(\frac{1+\lambda}{2}+\nu\Big)\, \Gamma\Big(\frac{1+\lambda}{2}-\nu\Big) \tag{A.3}$$

Some functional relations for the gamma-function (duplication and supplement) have been applied. The integrals (A.1)–(A.3) are absolutely convergent within the region $2\,|\mathrm{Re}\,\nu| - 1 < \mathrm{Re}\,\lambda < 0$. The left inequality stems from the behaviour of the integrand for small x and secures infrared convergence. From the asymptotic behaviour of the Hankel functions the right inequality follows, which stands for ultraviolet convergence. The integrals (A.2) and (A.3) are convergent even for $\mathrm{Re}\,\lambda < 1$ due to the oscillating behaviour of the Hankel functions.

In section 3 we need the following integral of a product of four Hankel functions:

$$\mathcal{I} \;:=\; \int_0^\infty dx\, x^{\lambda-1} H_\nu^{(1)^2}(ax)\, H_\nu^{(2)^2}(x)\;, \quad 0 < a \leq 1 \tag{A.4}$$

Such an integral of four Bessel functions was not found in the mathematical standard literature. Hence it will be evaluated explicitly now. The integrand behaves like $x^{\lambda-1-4|\mathrm{Re}\,\nu|}$ for $x \to 0$ and like $x^{\lambda-3}$ for $x \to \infty$, so that the integral (A.4) is absolutely convergent in the region $4\,|\mathrm{Re}\,\nu| < \mathrm{Re}\,\lambda < 2$. If $a \neq 1$ it is convergent even for $\mathrm{Re}\,\lambda < 3$ due to the oscillating behaviour of the integrant. We suppose that we are within this region since then all integrals and series appearing in the

following will be convergent, too.

For the square of $H_\nu^{(1)}(x)$ we use an integral representation:

$$H_\nu^{(1)^2}(x) \;=\; \frac{-2}{\pi^{\frac{3}{2}}}\, e^{-i\pi\nu} \int\limits_{-i\infty-\alpha}^{+i\infty-\alpha} \frac{ds}{2\pi i}\, x^{2s} e^{-i\pi s} \frac{\Gamma(-s)\,\Gamma(-s-\nu)\,\Gamma(-s+\nu)}{\Gamma(\frac{1}{2}-s)} \tag{A.5}$$

$\forall\, x \in \mathbb{R}^+$ and $|\mathrm{Re}\,\nu| < \alpha < \frac{3}{4}$. Integrals of this type are called Mellin-Barnes integrals. In order to prove (A.5) we close the contour of integration by a sequence of semicircles in the right half complex plane extended by α to the left with their centres in the origin according to figure A.1. The radii of these semicircles have to be choosen in such a way that they do not come too close to the poles of the three gamma-functions in the numerator. Using functional relations and Stirling's asymptotic expansion for the gamma-function one shows similar to [15, §14.5] that the contributions of the semicircles to the integral are vanishing in the limit of an

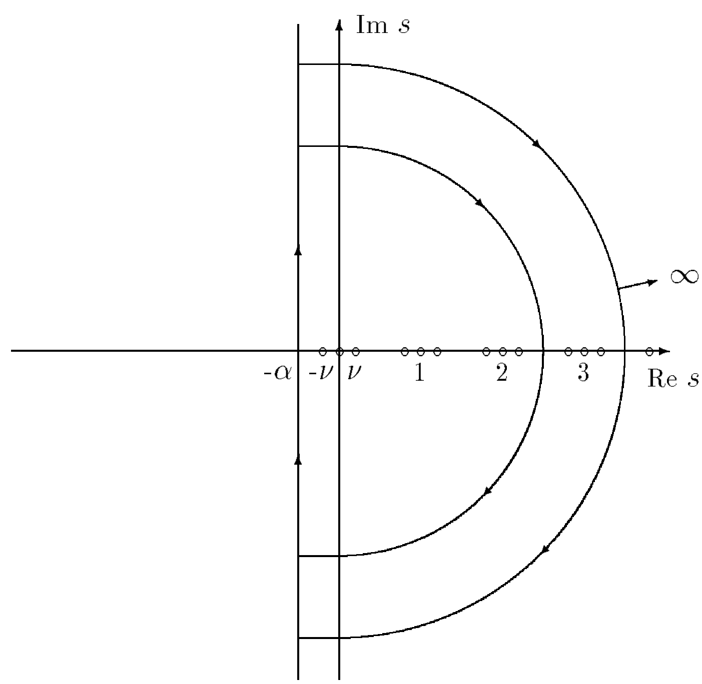

Figure A.1: The contour of integration in the complex s-plane is being closed by a sequence of semicircles, whose radii tend to infinity. The poles of the integrand in (A.5) are marked on the real axis.

infinite radius. Thus the theorem of residues can be used to evaluate the integral (A.5). The gamma-function $\Gamma(s)$ has simple poles at all negative integers $s = -n$, $n \in \mathbb{N}_0$ with the residues $(-)^n/n!$. The integral (A.5) becomes the sum of the residues of all poles of the three gamma-functions in the numerator of the integrand:

$$\begin{aligned} H_\nu^{(1)^2}(x) &= \frac{-2}{\pi^{\frac{3}{2}}} e^{-i\pi\nu} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \bigg(x^{2n} e^{-i\pi n} \frac{\Gamma(-n+\nu)\,\Gamma(-n-\nu)}{\Gamma(\frac{1}{2}-n)} \\ &\qquad + x^{2(n-\nu)} e^{-i\pi(n-\nu)} \frac{\Gamma(-n+\nu)\,\Gamma(-n+2\nu)}{\Gamma(\frac{1}{2}-n+\nu)} \\ &\qquad + x^{2(n+\nu)} e^{-i\pi(n+\nu)} \frac{\Gamma(-n-\nu)\,\Gamma(-n-2\nu)}{\Gamma(\frac{1}{2}-n-\nu)} \bigg) \end{aligned}$$

Via functional relations for the gamma-function and the power series representation [20, §5.41] of the product of two Bessel functions J_ν the proof of (A.5) can be completed:

$$\begin{aligned} H_\nu^{(1)^2}(x) &= \frac{-1}{\sin^2 \nu\pi} \bigg(-2\, e^{-i\pi\nu} \sum_{n=0}^{\infty} \frac{(-)^n\; \Gamma(1+2n)\; (x/2)^{2n}}{n!\,\Gamma(1-\nu+n)\,\Gamma(1+\nu+n)\,\Gamma(1+n)} \\ &\qquad + \sum_{n=0}^{\infty} \frac{(-)^n\; \Gamma(1-2\nu+2n)\; (x/2)^{2n-2\nu}}{n!\,\Gamma(1-\nu+n)\,\Gamma(1-2\nu+n)\,\Gamma(1-\nu+n)} \\ &\qquad + e^{-2i\pi\nu} \sum_{n=0}^{\infty} \frac{(-)^n\; \Gamma(1+2\nu+2n)\; (x/2)^{2n+2\nu}}{n!\,\Gamma(1+\nu+n)\,\Gamma(1+2\nu+n)\,\Gamma(1+\nu+n)} \bigg) \\ &= \frac{-1}{\sin^2 \nu\pi} \left(-2\, e^{-i\pi\nu}\, J_\nu J_{-\nu} + J_{-\nu}^2 + e^{-2i\pi\nu}\, J_\nu^2 \right) \qquad \text{q.e.d.} \end{aligned}$$

With the integral representation (A.5) our integral (A.4) reads after interchanging the order of integration:

$$\mathcal{I} = \frac{-2}{\pi^{\frac{3}{2}}} e^{-i\pi\nu} \int\limits_{-i\infty-\alpha}^{+i\infty-\alpha} \frac{ds}{2\pi i}\, a^{2s} e^{-i\pi s} \frac{\Gamma(-s)\,\Gamma(-s-\nu)\,\Gamma(-s+\nu)}{\Gamma(\frac{1}{2}-s)} \int\limits_0^{\infty} dx\, x^{\lambda-1+2s}\, H_\nu^{(2)^2}(x)$$

Choosing α such that $\operatorname{Re}\lambda - 2 < 2\alpha < \operatorname{Re}\lambda - 2\,|\operatorname{Re}\nu|$ we put in the Weber-Schafheitlin integral (A.3) and obtain:

$$\begin{aligned} \mathcal{I} &= \frac{2}{\pi^3} (-i)^\lambda \int\limits_{-i\infty-\alpha}^{+i\infty-\alpha} \frac{ds}{2\pi i} \left(a^2 e^{-2\pi i}\right)^s \\ &\quad \cdot \frac{\Gamma(-s)\,\Gamma(-s-\nu)\,\Gamma(-s+\nu)}{\Gamma(\frac{1}{2}-s)} \frac{\Gamma(\frac{\lambda}{2}+s)\,\Gamma(\frac{\lambda}{2}+s+\nu)\,\Gamma(\frac{\lambda}{2}+s-\nu)}{\Gamma(\frac{\lambda+1}{2}+s)} \end{aligned} \tag{A.6}$$

For $a \leq 1$ we can close the contour of integration again by a sequence of extended semicircles in the right half complex plane with their radii tending to infinity (figure A.1). From the theorem of residues we find:

$$\begin{aligned}
\mathcal{I} &= \frac{(-i)^\lambda}{\pi^2 \sin^2 \nu\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\lambda}{2}+n)}{n!} \Big(-2\, a^{2n} \frac{\Gamma(\frac{\lambda}{2}-\nu+n)\,\Gamma(\frac{\lambda}{2}+\nu+n)\,\Gamma(\frac{1}{2}+n)}{\Gamma(1+\nu+n)\,\Gamma(1-\nu+n)\,\Gamma(\frac{\lambda+1}{2}+n)} \\
&\qquad + a^{2n-2\nu} e^{2\pi i\nu} \frac{\Gamma(\frac{\lambda}{2}-\nu+n)\,\Gamma(\frac{\lambda}{2}-2\nu+n)\,\Gamma(\frac{1}{2}-\nu+n)}{\Gamma(1-\nu+n)\,\Gamma(1-2\nu+n)\,\Gamma(\frac{\lambda+1}{2}-\nu+n)} \\
&\qquad + a^{2n+2\nu} e^{-2\pi i\nu} \frac{\Gamma(\frac{\lambda}{2}+\nu+n)\,\Gamma(\frac{\lambda}{2}+2\nu+n)\,\Gamma(\frac{1}{2}+\nu+n)}{\Gamma(1+\nu+n)\,\Gamma(1+2\nu+n)\,\Gamma(\frac{\lambda+1}{2}+\nu+n)} \Big) \\
&= \frac{(-i)^\lambda}{\pi^2 \sin^2 \nu\pi} \Big(-2\, {}_4\bar{F}_3\big(\frac{\lambda}{2}, \frac{\lambda}{2}-\nu, \frac{\lambda}{2}+\nu, \frac{1}{2}; 1+\nu, 1-\nu, \frac{\lambda+1}{2}; a^2\big) \\
&\qquad + a^{-2\nu} e^{2\pi i\nu}\, {}_4\bar{F}_3\big(\frac{\lambda}{2}-\nu, \frac{\lambda}{2}-2\nu, \frac{\lambda}{2}, \frac{1}{2}-\nu; 1-\nu, 1-2\nu, \frac{\lambda+1}{2}-\nu; a^2\big) \\
&\qquad + a^{2\nu} e^{-2\pi i\nu}\, {}_4\bar{F}_3\big(\frac{\lambda}{2}+\nu, \frac{\lambda}{2}, \frac{\lambda}{2}+2\nu, \frac{1}{2}+\nu; 1+\nu, 1+2\nu, \frac{\lambda+1}{2}+\nu; a^2\big)\Big) \qquad \text{(A.7)}
\end{aligned}$$

We have applied functional relations and introduced our generalized hypergeometric function

$$\begin{aligned}
{}_p\bar{F}_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) &:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\ \cdots\ \Gamma(\alpha_p+n)}{n!\,\Gamma(\beta_1+n)\ \cdots\ \Gamma(\beta_q+n)}\ z^n \qquad \text{(A.8)} \\
&= \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}\, {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),
\end{aligned}$$

where ${}_pF_q$ is the usual generalized hypergeometric function.

Note that (A.6) is an integral of the Mellin-Barnes type, too, and is therefore a special case of the general notion of Meier's G-function [16]:

$$\mathcal{I} = \frac{2}{\pi^3} (-i)^\lambda\, G^{33}_{44} \left(a^2 e^{-2\pi i} \,\middle|\, \begin{matrix} 1-\frac{\lambda}{2}, 1-\frac{\lambda}{2}+\nu, 1-\frac{\lambda}{2}-\nu, & \frac{1}{2} \\ 0\ ,\quad \nu\quad ,\quad -\nu\quad , & \frac{1-\lambda}{2} \end{matrix} \right)$$

In section 3 we have $\lambda = d+l$ and $a = e^{-t}$, and it turns out to be reasonable to introduce the following function G:

$$\begin{aligned}
G(t;p,l) &:= \frac{-e^{-td}}{4\sin^2 \pi\nu} \Big(-2\, {}_4\bar{F}_3\big(\frac{d+l}{2}-p, \frac{d+l}{2}-\nu, \frac{d+l}{2}+\nu, \frac{1}{2}; 1+\nu, 1-\nu, \frac{d+l+1}{2}; e^{-2t}\big) \\
&\quad + e^{2\nu t}\, e^{2\pi i\nu}\, {}_4\bar{F}_3\big(\frac{d+l}{2}-\nu-p, \frac{d+l}{2}-2\nu, \frac{d+l}{2}, \frac{1}{2}-\nu; 1-\nu, 1-2\nu, \frac{d+l+1}{2}-\nu; e^{-2t}\big) \\
&\quad + e^{-2\nu t}\, e^{-2\pi i\nu}\, {}_4\bar{F}_3\big(\frac{d+l}{2}+\nu-p, \frac{d+l}{2}, \frac{d+l}{2}+2\nu, \frac{1}{2}+\nu; 1+\nu, 1+2\nu, \frac{d+l+1}{2}+\nu; e^{-2t}\big)\Big)
\end{aligned}$$

(A.9)

Then our integral (A.4) reads just

$$e^{-td}\,\mathcal{I} \;=\; -\,\frac{4}{\pi^2}\,(-i)^{d+l}\,G(t;0,l)\,. \tag{A.10}$$

With the auxiliary variable p it is possible to express the time derivative of a G-function in terms of G-functions:

$$\left(\frac{1}{2}\,\partial_t + p + 1 - \frac{l}{2}\right) G(t;p{+}1,l) \;=\; -\,G(t;p,l) \tag{A.11}$$

This relation can simply be proven by substituting the definitions (A.9) and (A.8).

B Geometrical tensors for the k=0 FRW-metric

The k=0 FRW-metric

$$g_{00} \;=\; 1\,, \qquad g_{0i} \;=\; 0\,, \qquad g_{ij} \;=\; -a^2(t)\,\delta_{ij} \tag{B.1}$$

leads to the Christoffel symbols

$$\Gamma^0_{ij} \;=\; H\,a^2\delta_{ij}\,, \qquad \Gamma^i_{j0} \;=\; \Gamma^i_{0j} \;=\; H\,\delta^i_j\,, \qquad 0 \text{ otherwise}\,, \quad H = \dot{a}/a\,. \tag{B.2}$$

From them we obtain the Riemann tensor

$$\begin{aligned}
R^\mu{}_{\nu\rho\sigma} &:= \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \;: \\
R^0{}_{i0j} &= -R^0{}_{ij0} \;=\; a^2(\dot{H} + H^2)\,\delta_{ij}\,, \quad R^i{}_{00j} \;=\; -R^i{}_{0j0} \;=\; (\dot{H} + H^2)\,\delta^i_j\,, \\
R^i{}_{kjl} &= a^2H^2(\delta^i_j\delta_{kl} - \delta^i_l\delta_{kj})\,, \qquad 0 \text{ otherwise}\,,
\end{aligned} \tag{B.3}$$

as well as the Ricci tensor and curvature scalar

$$\begin{aligned}
R_{\mu\nu} &:= R^\lambda{}_{\mu\lambda\nu} \;: \quad R_{00} \;=\; -d\,(\dot{H} + H^2)\,, \quad R_{ij} \;=\; (\dot{H} + d\,H^2)\,a^2\delta_{ij}\,, \quad R_{0i} \;=\; 0 \\
R &:= g^{\mu\nu}R_{\mu\nu} \;=\; -d\,(2\dot{H} + (d{+}1)\,H^2)\,.
\end{aligned} \tag{B.4}$$

The H-tensors read:

$$
\begin{aligned}
H_{\mu\nu} &:= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{d+1}x\,\sqrt{|g|}\,R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma} \\
&= -\frac{1}{2}g_{\mu\nu}R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}+2R_{\mu\alpha\beta\rho}R_\nu{}^{\alpha\beta\rho}+4\Box R_{\mu\nu}-2R_{;\mu\nu}-4R_{\mu\alpha}R^\alpha{}_\nu \\
&\quad +4R^{\alpha\beta}R_{\alpha\mu\beta\nu} \\
{}^{(1)}H_{\mu\nu} &:= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{d+1}x\,\sqrt{|g|}\,R^2 \\
&= -\frac{1}{2}g_{\mu\nu}R^2+2R\,R_{\mu\nu}-2R_{;\mu\nu}+2g_{\mu\nu}\Box R \\
{}^{(2)}H_{\mu\nu} &:= \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^{d+1}x\,\sqrt{|g|}\,R^{\alpha\beta}R_{\alpha\beta} \\
&= -\frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}+2R_\mu{}^\alpha R_{\alpha\nu}-2R_\mu{}^\alpha{}_{;\nu\alpha}+\Box R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\Box R \qquad \text{(B.5)}
\end{aligned}
$$

Their non-zero components are:

$$
\begin{aligned}
H_{00} &= d\,(-4H\ddot{H}+2\dot{H}^2-4dH^2\dot{H}+(3-d)H^4) \\
H_{ij} &= a^2\delta_{ij}\,(4\dddot{H}+8dH\ddot{H}+6d\dot{H}^2+4(d^2+d-3)H^2\dot{H}+d(d-3)H^4) \\
{}^{(1)}H_{00} &= d^2(-4H\ddot{H}+2\dot{H}^2-4dH^2\dot{H}-\tfrac{1}{2}(d+1)(d-3)H^4) \\
{}^{(1)}H_{ij} &= a^2\delta_{ij}\,d(4\dddot{H}+8dH\ddot{H}+6d\dot{H}^2+(6d^2-4d-6)H^2\dot{H}+\tfrac{1}{2}d(d+1)(d-3)H^4) \\
{}^{(2)}H_{00} &= d(-(d+1)H\ddot{H}+\tfrac{1}{2}(d+1)\dot{H}^2-d(d+1)H^2\dot{H}+\tfrac{1}{2}d(3-d)H^4) \\
{}^{(2)}H_{ij} &= a^2\delta_{ij}\,((d+1)\dddot{H}+2d(d+1)H\ddot{H}+\tfrac{3}{2}d(d+1)\dot{H}^2+d(d^2+3d-6)H^2\dot{H} \\
&\quad +\ \tfrac{1}{2}d^2(d-3)H^4) \qquad \text{(B.6)}
\end{aligned}
$$

The Weyl tensor vanishes for the conformal flat FRW-metric, so that the relation $d(d-1)\,H_{\mu\nu}+2\,{}^{(1)}H_{\mu\nu}-4d\,{}^{(2)}H_{\mu\nu}=0$ results. In $d+1=4$ spacetime dimensions the Euler number $n=\int d^4x\,|g|^{1/2}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+R^2-4\,R_{\mu\nu}R^{\mu\nu})$ is a topological invariant, hence its variational derivative $H_{\mu\nu}+{}^{(1)}H_{\mu\nu}-4\,{}^{(2)}H_{\mu\nu}$ vanishes. The result (B.6) has been checked explicitly against these two identities.

For the de Sitter spacetime we have $\dot{H}=0$ and all H-tensors are vanishing for $d=3$.

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